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Nearby cycles and monodromy in étale cohomology

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Plan

1. Discs and traits
2. The functors $R\Psi$ and $R\Phi$
3. General theorems on nearby and vanishing cycles
4. Examples
5. Grothendieck's local monodromy theorem
6. The ℓ-adic weight spectral sequence
7. Further developments

1. Discs and traits

local ring A : henselian : any finite A-algebra $B =$ product of local rings
$\Leftrightarrow$ any *strictly essentially étale* local A-algebra is A-isomorphic to A (EGA IV 18.5.11, 18.6.6)

A strictly henselian : any *essentially étale* local A-algebra is A-isomorphic to A ($\Leftrightarrow$ henselian + residue field separably closed)

$\mathrm{Spec}\, A$ strictly local : A strictly henselian

complete noetherian local $\Rightarrow$ henselian trait : $S = \mathrm{Spec}\, R$, R a dvr (discrete valuation ring)

closed point : $s = \mathrm{Spec}\, k$, $k = R/\mathfrak{m}$
generic point $\eta = \mathrm{Spec}\, K$, $K = \mathrm{Frac}\, R$

S henselian, L/K finite separable $\Rightarrow O_L =$ product of finite local R-algebras

Analogies

Disc	Trait
open disc	strictly local trait
$D = \{\lvert z\rvert < 1\}$	$S = \operatorname{Spec} R$
$\{0\} \in D$	$s \in S$
$D^* = D - \{0\}$	$\eta \in S$
coordinate z in D	uniformizing parameter $\pi \in R$
$\widetilde{D^*} = \{\operatorname{Im} \tau > 0\} \to D^*$	$\overline{\eta} = \operatorname{Spec} \overline{K} \to \eta = \operatorname{Spec} K$
universal cover $\tau \mapsto \exp(2\pi i \tau)$	$\overline{K}$ = separable closure of K
$\pi_1(D^*, t) = \operatorname{Aut}(\widetilde{D^*}) = \mathbf{Z}$	inertia group $\pi_1(\overline{\eta}/\eta) = \operatorname{Gal}(\overline{K}/K)$

last analogy OK if $\operatorname{char}(k) = 0$, too coarse if $\operatorname{char}(k) = p > 0$

Stucture of inertia

$$I = \mathrm{Gal}(\overline{K}/K)$$

$$1 \to P \to I \xrightarrow{t} \widehat{\mathbf{Z}}'(1) \to 1$$

$$\widehat{\mathbf{Z}}'(1) = \mathrm{Gal}(\eta_t/\eta) = \varprojlim_{(n,p)=1} (\mathbf{Z}/n\mathbf{Z})(1)(k) = \prod_{\ell \neq p} \mathbf{Z}_\ell(1)(k)$$

$$\eta_t = \mathrm{Spec}\, K_t, \quad K_t = \cup_{(n,p)=1} K(\pi^{1/n}), \quad p = \mathrm{char.exp}(k)$$

(maximal tamely ramified extension of K in $\overline{K}$)

$t : I \to \widehat{\mathbf{Z}}'(1)$: tame character :

$$t(g) = g\pi'/\pi' \in \mu_n(k), \quad \pi'^n = \pi$$

$$1 \to P \to I \xrightarrow{t} \widehat{\mathbf{Z}}'(1) \to 1$$

Abhyankar's lemma $\Rightarrow$ P : a pro-p-group : wild inertia

(well understood if k alg. closed ; if not, complicated ramification (work in progress (Abbes, T. Saito, ...))

Arithmetic case

R henselian, k, K

$\overline{k}$: separable closure of k ; $R' = R^{sh}$ associated strict henselization, $R'/\mathfrak{m}' = \overline{k}$,

$K' = \mathrm{Frac}\, R' = K_{ur}$: maximal unramified extension of K

$$K \to K_{ur} \to (K_{ur})_t \to \overline{K}$$

($\overline{K}$ = separable closure of K_{ur})

$G_K = \mathrm{Gal}(\overline{K}/K)$, $G_k = \mathrm{Gal}(K_{ur}/K) = \mathrm{Gal}(\overline{k}/k), I = \mathrm{Gal}(K/K')$

$$\begin{tikzcd} 1 \arrow[r] & I \arrow[r] \arrow[d] & G_K \arrow[r] \arrow[d] & G_k \arrow[r] \arrow[d, "Id"] & 1 \\ 1 \arrow[r] & I_t \arrow[r] & (G_K)_t \arrow[r] & G_k \arrow[r] & 1 \end{tikzcd}$$

$I_t = \mathrm{Gal}(K'_t/K')$

G_k acts on I_t by conjugation , and isomorphism induced by tame character

$$t : I_t \simeq \widehat{\mathbf{Z}}'(1)$$

is G_k-equivariant : for $\sigma \in I_t$, $g \in G_k$,

$$t(g\sigma g^{-1}) = t(\sigma)^{\chi(g)}$$

where $\chi : G_k \to \widehat{\mathbf{Z}}'^{*}$ = cyclotomic character

Application : Grothendieck's local monodromy lemma

Lemma

Assume no finite extension of k contains all ℓ^n-th roots of 1 for $n \geq 1$. Let

$$\rho : G_K \to \mathrm{GL}(V)$$

be a continuous representation, V = finite dimensional $\mathbf{Q}_\ell$-vector space.
Then there exists an open subgroup $I' \subset I$ such that $\rho(\sigma)$ is unipotent for all $\sigma \in I'$.

Proof

Up to finite extension of K, WMA

$$(*) \qquad \mathrm{Im}(\rho) \subset 1 + \ell^2 M_n(\mathbf{Z}_\ell)$$

Will show : $\rho(\sigma)$ unipotent for all $\sigma \in I$

(*) $\Rightarrow \mathrm{Im}(\rho) =$ pro-ℓ-group

$\Rightarrow \rho$ factors through $\mathrm{Gal}(K_\ell/K)$, where $K_\ell = \cup K_{ur}(\pi^{1/\ell^n})$.

Let $\sigma \in I_\ell = \mathrm{Gal}(K_\ell/K_{ur}) = \mathbf{Z}_\ell(1)$.

Recall : for all $g \in G_k$,

$$(**) \qquad g\sigma g^{-1} = \sigma^{\chi(g)}$$

($\chi : G_k \to \mathbf{Z}_\ell^* =$ cyclotomic character)

$$(**) \qquad g\sigma g^{-1} = \sigma^{\chi(g)}$$

Let $x := \rho(\sigma)$,

$X = \log x = \sum_{n\geq 1}(-1)^{n-1}(x-1)^n/n \in \ell^2 M_n(\mathbf{Z}_\ell)$

(**) $\Rightarrow$

$$gXg^{-1} = \chi(g)X$$

$\Rightarrow$ for all $i \geq 1$

$$c_i(X) = \chi(g)^i c_i(X),$$

where

$$\det(X.Id - t) = X^n - c_1(X)X^{n-1} + \cdots + (-1)^n c_n(X).$$

Hypothesis on $k \Rightarrow \chi(G_k) = \mathrm{Gal}(k(\mu_{\ell^\infty})/k) \subset \mathbf{Z}_\ell^*$ infinite

$\Rightarrow$ there exists $g \in G_k$ s. t. $\chi(g)$ is of infinite order

since

$$c_i(X) = \chi(g)^i c_i(X),$$

get $c_i(X) = 0 \ \forall i \geq 1$

$\Rightarrow X$ nilpotent

$\Rightarrow$ (as $\ell \geq 2$), $x = \exp(\mathrm{X})$ ($\ell \geq 2$) unipotent. Qed.

Compare with Grothendieck's proof of ℓ-adic Chern classes of linear representations of discrete groups being torsion

(Th. 4.8 in [Classes de Chern et représentations ℓ-adiques des groupes discrets, Dix exposés sur la cohomologie des schémas, North Holland Pub. Co., 1968])

Theorem

(Grothendieck) G a discrete group of finite type, k separably closed, $\rho : G \to \mathrm{GL}(E)$, E/k finite dimensional, $\ell \neq \mathrm{char}(k)$. Then

$$c_i(\rho) \in H^{2i}(G, \mathbf{Z}_\ell(k)(i))$$

is torsion for all $i \geq 1$.

2. The functors $R\Psi$ and $R\Phi$

S = henselian trait

$\Lambda = \mathbf{Z}/\ell^{\nu}\mathbf{Z}$ (or finite over it) (or ℓ-adic variants)

X/S

$$X_{\bar{s}} \xrightarrow{\bar{i}} X_{S_{(\bar{s})}} \xleftarrow{\bar{j}} X_{\bar{\eta}}$$

over

$$\begin{tikzcd}
 & & \bar{\eta} \arrow[dl] \arrow[d] \\
\bar{s} \arrow[r] \arrow[d] & S_{(\bar{s})} \arrow[d] & \eta_{\mathrm{ur}} \arrow[l] \arrow[d] \\
s \arrow[r] & S & \eta \arrow[l]
\end{tikzcd}$$

$$F \in D^+(X_\eta, \Lambda)$$

$$X_{\bar{s}} \xrightarrow{\bar{i}} X_{S_{(\bar{s})}} \xleftarrow{\bar{j}} X_{\bar{\eta}}$$

$$R\Psi F := \bar{i}^* R\bar{j}_*(F|X_{\bar{\eta}}) \in D^+(X_{\bar{s}}, \Lambda)$$

nearby cycles complex (cf. [SGA 7 I, 2.2])

(Trivial) example : $X = S$, $R\Psi F = R\Gamma(S_{(\bar{s})}, R\bar{j}_* F_{\bar{\eta}}) = F_{\bar{\eta}}$

Alternate definition

(cf. [SGA 7 XIII 1.3])

$$
\begin{tikzcd}
X_{\widetilde{s}} \arrow[r, "\widetilde{i}"] \arrow[d] & X_{\widetilde{S}} \arrow[d] & X_{\overline{\eta}} \arrow[l, "\widetilde{j}"'] \arrow[dl, "\bar{j}"'] \\
X_{\overline{s}} \arrow[r, "\widetilde{i}"] & X_{S_{(\overline{s})}} &
\end{tikzcd}
$$

($\widetilde{S}$ = integral closure of $S_{(\overline{s})}$ in $\overline{\eta}$, $\widetilde{s} \to \overline{s}$ radicial)

$$R\Psi F = \widetilde{i}^* R\widetilde{j}_*(F|X_{\overline{\eta}})$$

(use pbc and $(X_{\overline{s}})_{\mathrm{et}} = (X_{\widetilde{s}})_{\mathrm{et}}$)

Stalks

$\overline{x} \to X_{\overline{s}}$ geometric point

Milnor ball $X_{(\overline{x})}$ (strict localization at $\overline{x}$)

Milnor fiber $(X_{(\overline{x})})_{\overline{\eta}}$

$$(R\Psi F)_{\overline{x}} = R\Gamma((X_{(\overline{x})})_{\overline{\eta}}, F)$$

Galois action

$G = \mathrm{Gal}(\overline{\eta}/\eta)$ acts on $R\Psi F$:

$R\Psi F$ underlies a complex of sheaves (of Λ-modules) on $X_{\overline{s}}$ with a continuous action of G
compatible with action of G on $X_{\overline{s}}$ via $G \to \mathrm{Gal}(\overline{s}/s)$

Define

$X_s \overleftarrow{\times}_S \eta$:= topos of G-sheaves on $X_{\overline{s}}$ (Deligne's oriented product)

Then :

$$R\Psi F \in D^+(X_s \overleftarrow{\times}_S \eta, \Lambda)$$

In particular :

- $R^q\Psi F$ is a G-sheaf on $X_{\overline{s}}$
- any $g \in G$ defines an automorphism

$$g^* \in \mathrm{Aut}(R\Psi F)$$

of the underlying object $R\Psi F$ of $D^+(X_{\overline{s}}, \Lambda)$

Vanishing cycles

For $K \in D^+(X, \Lambda)$ (not in $D^+(X_\eta, \Lambda)$, adjunction map defines G-equivariant triangle (i. e. a triangle of $D^+(X_s \overset{\leftarrow}{\times}_S \eta, \Lambda)$)

$$K|X_{\bar{s}} \to R\Psi(K|X_\eta) \to R\Phi K \to$$

$R\Phi K$: vanishing cycles complex

Trivial example (cont'd) $X = S$, $K \in D^+(S, \Lambda)$

$$K_{\bar{s}} \xrightarrow{\mathrm{sp}} K_{\bar{\eta}} \to R\Phi K \to$$

sp = specialization map

$(X/S, K)$ locally acyclic at $x \in X_s \Leftrightarrow_{\mathrm{def}} (R\Phi K)_{\overline{x}} = 0$ ($\overline{x} \to x$ geometric point)

locally acyclic $\Leftrightarrow_{\mathrm{def}}$ locally acyclic at each point

$\Leftrightarrow K_{\overline{x}} \xrightarrow{\sim} R\Gamma((X_{(\overline{x})})_{\overline{\eta}}, K)$

X/S smooth, ℓ invertible on S, K a lisse sheaf

$\Rightarrow (X/S, K)$ locally acyclic (Artin's local acyclicity theorem)

But if $\ell = p$, $p = \mathrm{char}(k)$, X/S smooth, $\Lambda = \mathbf{Z}/p^n\mathbf{Z}$,

$R\Phi K$ highly non-trivial (studied by Bloch-Kato, Tsuji, ...)

Invariants under inertia, tame nearby cycles

$F \in D^+(X_\eta, \Lambda)$

$$\begin{tikzcd} & & X_{\overline{\eta}} \arrow[dl, "\bar{j}"'] \arrow[d] & \overline{\eta} \arrow[d, "I"'] \\ X_{\bar{s}} \arrow[r, "\bar{i}"] & X_{S_{(\bar{s})}} & X_{\eta_{\mathrm{ur}}} \arrow[l, "j_{ur}"'] & \eta_{\mathrm{ur}} \end{tikzcd}$$

$$\bar{i}^* R j_{\mathrm{ur}*}(F | X_{\mathrm{ur}}) = R\Gamma(I, R\Psi F)$$

$$1 \to P \to I \xrightarrow{t} I_{\mathrm{t}} \to 1$$

P a pro-p-group (wild inertia), $p = \mathrm{char.exp}(k)$, $t =$ tame character,

$I_{\mathrm{t}} = \prod_{\ell' \neq p} \mathbf{Z}_{\ell'}(1)$

$$1 \to P \to I \xrightarrow{t} I_{\mathrm{t}} \to 1$$

$$\begin{tikzcd}
 & & X_{\overline{\eta}} \arrow[d] \arrow[ddl, "\bar{j}"'] & \overline{\eta} \arrow[d, "P"'] \\
 & & X_{\eta_{\mathrm{t}}} \arrow[d] \arrow[dl, "j_{\mathrm{t}}"'] & \eta_{\mathrm{t}} \arrow[d, "I_{\mathrm{t}}"'] \\
X_{\bar{s}} \arrow[r, "\bar{i}"] & X_{S_{(\bar{s})}} & X_{\eta_{\mathrm{ur}}} \arrow[l, "j_{ur}"'] & \eta_{\mathrm{ur}}
\end{tikzcd}$$

Tame nearby cycles

$$R\Psi_{\mathrm{t}}F := R\Gamma(P, R\Psi F) = \bar{i}^* Rj_{\mathrm{t}*}(F|X_{\eta_{\mathrm{t}}})$$

$R\Psi F$ tame $\Leftrightarrow_{\mathrm{def}}$ $R\Psi_{\mathrm{t}}F \xrightarrow{\sim} R\Psi F$ (I acts through I_{t})

Note : $M \mapsto \Gamma(P, M) = M^P$ exact on Λ-modules if $\ell \neq p$

3. General theorems on nearby and vanishing cycles

3.1. Functoriality

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & Y \arrow[dl, "g"] \\ S & \end{tikzcd} \qquad \begin{tikzcd} X_{\bar{s}} \arrow[r, "\bar{i}"] \arrow[d, "h_{\bar{s}}"'] & X_{S_{(\bar{s})}} \arrow[d, "h"] & X_{\bar{\eta}} \arrow[l, "\bar{j}"'] \arrow[d, "h_{\bar{\eta}}"] \\ Y_{\bar{s}} \arrow[r, "\bar{i}"] & Y_{S_{(\bar{s})}} & Y_{\bar{\eta}} \arrow[l, "\bar{j}"'] \end{tikzcd}$$

(1) Push-out

bc map (for Rh_*) gives

$$R\Psi Rh_{\eta *}F \to Rh_{\bar{s}*}R\Psi F$$

isomorphism if h proper (pbc)

In particular ($Y = S$), if X/S proper, get G-equivariant isomorphism

$$R\Gamma(X_{\overline{\eta}}, F) \xrightarrow{\sim} R\Gamma(X_{\overline{s}}, R\Psi F)$$

and, for $K \in D^+(X, \Lambda)$, long exact sequence

$$\cdots \to H^i(X_{\overline{s}}, K|X_{\overline{s}}) \xrightarrow{\mathrm{sp}} H^i(X_{\overline{\eta}}, K|X_{\overline{\eta}}) \to H^i(X_{\overline{s}}, R\Phi K) \to \cdots$$

$$\mathrm{sp} : H^i(X_{\overline{s}}, K|X_{\overline{s}}) \xleftarrow{\sim} H^i(X_{S_{(\overline{s})}}, K) \to H^i(X_{\overline{\eta}}, K|X_{\overline{\eta}})$$

called specialization map

$R\Gamma(X_{\overline{s}}, R\Phi K)$ measures defect of sp being an isomorphism

(X, K) locally acyclic outside closed $\Sigma \subset X_s$

$\Rightarrow$ defect concentrated on Σ

X/S proper and smooth, $\ell \neq p \Rightarrow \mathrm{sp} : R\Gamma(X_s, \Lambda) \xrightarrow{\sim} R\Gamma(X_{\overline{\eta}}, \Lambda)$

NB. Fails for $\ell = p$ (first case : jump of p-rank of elliptic curve)

(2) Pull-back

$$\begin{tikzcd} X_{\bar{s}} \arrow[r, "\bar{i}"] \arrow[d, "h_{\bar{s}}"'] & X_{S_{(\bar{s})}} \arrow[d, "h"'] & X_{\overline{\eta}} \arrow[l, "\bar{j}"'] \arrow[d, "h_{\overline{\eta}}"'] \\ Y_{\bar{s}} \arrow[r, "\bar{i}"] & Y_{S_{(\bar{s})}} & Y_{\overline{\eta}} \arrow[l, "\bar{j}"'] \end{tikzcd}$$

bc map (for $R\bar{j}_*$) gives

$$h_{\bar{s}}^* R\Psi F \to R\Psi h_{\eta}^* F$$

isomorphism if ℓ invertible on S and h smooth (smooth bc)

(fails for h smooth but $\ell = p$)

From now on we assume ℓ invertible on S

3.2. Base change

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd} \qquad \begin{tikzcd} X'_{\overline{s}'} \arrow[r] \arrow[d] & X_{\overline{s}} \arrow[d] \\ \overline{s}' \arrow[r] & \overline{s} \end{tikzcd}$$

cartesian squares, with $S' \to S$ dominant map of henselian traits,

$F \in D^+(X_\eta, \Lambda)$. Then, bc map

$$(R\Psi_{X/S}F)|X'_{\overline{s}'} \to R\Psi_{X'/S'}(F|X'_{\eta'})$$

is an isomorphism (Deligne, SGA 4 1/2, Th. finitude 3.7)

(trivial if S' = normalization of S in finite extension of $k(\eta)$ contained in $k(\overline{\eta})$)

3.3. Finiteness

$D_c^+(T, \Lambda) := \{K \in D^+(T, \Lambda) | \mathcal{H}^q(K) \text{ constructible} \, \forall q\}$
(ditto for D_c^b)

Assume X/S of finite type. Then :

$$R\Psi : D_c^+(X_\eta, \Lambda) \to D_c^+(X_{\overline{s}}, \Lambda)$$

(Deligne, SGA 4 1/2 Th. finitude, 3.2)

Moreover, affine Lefschetz (Artin) (SGA 4 XIV) implies :

$$R^q \Psi F = 0$$

for $q > \dim(X_\eta)$

and all sheaves of Λ-modules F on X_η (SGA 7 I 4.2)

In particular

$$R\Psi : D^b_c(X_\eta, \Lambda) \to D^b_c(X_{\overline{s}}, \Lambda)$$

and

$$R\Psi : D_{ctf}(X_\eta, \Lambda) \to D_{ctf}(X_{\overline{s}}, \Lambda)$$

where

$$D_{ctf}(T, \Lambda) = \{K \in D^b_c(T, \Lambda) | K \text{ of finite tor-dimension } \}$$

(D_{ctf} important for ℓ-adic formalism :

roughly, $D^b_c(T, \mathbf{Z}_\ell) = 2 - \varprojlim D_{ctf}(T, \mathbf{Z}/\ell^n\mathbf{Z})$)

3.4. Duality and perversity

Recall biduality : for T regular noetherian of dimension 0 or 1, and $a : Z \to T$,

$K_Z := Ra^! \Lambda_T$ is dualizing,

i. e. $D_Z := R\mathcal{H}om(-, K_Z)$ sends D^b_c to D^b_c and $D_Z D_Z = Id$ (Deligne, SGA 4 1/2, Th. finitude, 4.3)

Assume $f : X \to S$ separated and of finite type.

Theorem (Gabber, [I] 4.2) For $F \in D^b_c(X_\eta, \Lambda)$, have canonical isomorphism of $D^b_c(X_s \overset{\leftarrow}{\times}_S \eta, \Lambda)$ (i. e. "G-equivariant in $D^b_c(X_{\bar{s}}, \Lambda)$")

$$R\Psi D_{X_\eta} F \xrightarrow{\sim} D_{X_{\bar{s}}} R\Psi F$$

I = L. I., Autour du théorème de monodromie locale, in Astérisque 223

$$(*) \qquad R\Psi D_{X_\eta} F \xrightarrow{\sim} D_{X_{\bar{s}}} R\Psi F$$

Corollary 1

$$R\Psi : \mathrm{Per}(X_\eta, \Lambda) \to \mathrm{Per}(X_{\bar{s}}, \Lambda)$$

where $\mathrm{Per}(-, \tilde{ })$ = full subcategory of $D^b_c(-, \Lambda)$ consisting of perverse sheaves,

Here $\Lambda = \mathbf{Z}/\ell^\nu\mathbf{Z}$, or finite extension of $\mathbf{Q}_\ell$, or $\overline{\mathbf{Q}}_\ell$ (complications for $\mathbf{Z}_\ell$-coefficients)

Proof of corollary 1 : $R\Psi$ right t-exact ([BBD],4.4.2) (i. e. preserves ${}^pD^{\leq 0}$);

(*) $\Rightarrow$ $R\Psi$ left t-exact, hence t-exact

Recall : T separated, finite over a field k, ℓ invertible in k, then, for $K \in D^b_c(T,\Lambda)$

$$K \in {}^pD^{\leq 0} \Leftrightarrow \mathcal{H}^q i_x^* K = 0 \, \forall q > -\dim(x)$$

$$K \in {}^pD^{\geq 0} \Leftrightarrow \mathcal{H}^q i_x^! K = 0 \, \forall q < -\dim(x)$$

$$\mathrm{Per}(T,\Lambda) = {}^pD^{\leq 0}(T,\Lambda) \cap {}^pD^{\geq 0}(T,\Lambda)$$

right t-exact : sends ${}^pD^{\leq 0}$ to ${}^pD^{\leq 0}$

left t-exact : sends ${}^pD^{\geq 0}$ to ${}^pD^{\geq 0}$

t-exact : both left and right t-exact (hence sends Per to Per)

Recall triangle defining vanishing cycles :

$$K|X_{\bar{s}} \to R\Psi(K|X_\eta) \to R\Phi K \to$$

Corollary 2 (Gabber [I] 4.6)

$$K \in \mathrm{Per}(X, \Lambda) \Rightarrow R\Phi K[-1] \in \mathrm{Per}(X_{\bar{s}}, \Lambda)$$

X/S, S a trait, not a field, t-structure on $D^b(X, \Lambda)$ defined by :

$$K \in {}^pD^{\leq 0}(X, \Lambda) \Leftrightarrow j^*K \in {}^pD^{\leq -1}(X_\eta, \Lambda) \,\text{and}\, i^*K \in {}^pD^{\leq 0}(X_s, \Lambda)$$

$$K \in {}^pD^{\geq 0}(X, \Lambda) \Leftrightarrow j^*K \in {}^pD^{\geq -1}(X_\eta, \Lambda) \,\text{and}\, i^*K \in {}^pD^{\geq 0}(X_s, \Lambda)$$

3.5. Künneth

X_i/S finite type (i = 1,2), $X := X_1 \times_S X_2$, $F_i \in D_{ctf}((X_i)_\eta, \Lambda)$

$F = F_1 \boxtimes^L F_2 := \mathrm{pr}_1^* F_1 \otimes^L \mathrm{pr}_2^* F_2$.

$$\begin{tikzcd} & X \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ X_1 \arrow[dr] & & X_2 \arrow[dl] \\ & S & \end{tikzcd}$$

Theorem (Gabber, [I] 4.7) : The Künneth map

$$R\Psi_{X_1/S} F_1 \boxtimes^L R\Psi_{X_2/S} F_2 \to R\Psi_{X/S} F$$

is an isomorphism (of $D(X_s \overleftarrow{\times}_S \eta, \Lambda)$). (This is not formal.)

Indications on proofs of 3.2 to 3.5

• Deligne's method (SGA 4 1/2, Th. finitude) : use induction on dimension, cut out by pencils, concentrate the defect on a finite number of closed points, conclude by a global argument

• alternate method : use dévissage and de Jong's alterations to reduce to the semistable reduction case, treated by direct calculation (see § 4)

3.6. Comparison with complex nearby cycles

Recall : $X/\mathbf{C}$ loc. finite type $\mapsto$ analytic space X_{cl} ($= X(\mathbf{C})$, classical topology : usual, or local isomorphisms)

étale map $X \to Y$ gives local isomorphism $X_{cl} \to Y_{cl}$, hence we have a canonical map

$$\varepsilon : X_{cl} \to X_{et}$$

$(\varepsilon^*(U) = U_{cl})$

$\Lambda = \mathbf{Z}/N\mathbf{Z}$; for $F \in D^+(X_{et}, \Lambda)$, get a comparison map

$$(*) \qquad R\Gamma(X_{et}, F) \to R\Gamma(X_{cl}, \varepsilon^* F)$$

Theorem

(Artin) For $X/\mathbf{C}$ finite type and $F \in D_c^+(X_{et}, \Lambda)$ (i. e. $\mathcal{H}^q F$ constructible for all q), (*) = isomorphism

$$\varepsilon : X_{cl} \to X_{et}$$

$$(*) \qquad R\Gamma(X_{et}, F) \xrightarrow{\sim} R\Gamma(X_{cl}, \varepsilon^* F)$$

Generalization for $f : X \to Y$ finite type :

$$\varepsilon^* R f_{et*} F \xrightarrow{\sim} R f_{cl*}(\varepsilon^* F)$$

$(F \in D_c^+(X, \Lambda))$

Comparison between $R\Psi_{et}$ and $R\Psi_{cl}$

Set-up : $Y/\mathbf{C}$ smooth connected curve, $0 \in Y(\mathbf{C})$, $f : X \to Y$ separated, finite type, $X_0 = f^{-1}(0)$

- $R\Psi (= R\Psi_{et})$

  S : henselization of Y at 0, $0 \to S \leftarrow \eta \leftarrow \overline{\eta} = \varprojlim \eta(t^{1/n})$

$$R\Psi : D^+(X - X_0, \Lambda) \to D^+(X_0, \Lambda)$$

+ action of $G = \mathrm{Gal}(\overline{\eta}/\eta)(\overset{\sim}{\to} \widehat{\mathbf{Z}}(1))$ on $R\Psi F$

$$R\Psi : D^+(X - X_0, \Lambda) \to D^+(X_0 \times \eta, \Lambda)$$

$(\mathrm{Sh}(X_0 \times \eta, \Lambda) =$ sheaves of $\Lambda[G]$-modules on X_0 $(\overset{\sim}{\to} (X_0)_{et} \times B\widehat{\mathbf{Z}}(1))$

$$R\Psi K = i^* R\bar{j}_*(K|X_{\overline{\eta}}),$$

$$X_0 \xrightarrow{i} X_S \xleftarrow{\bar{j}} X_{\overline{\eta}}$$

- $R\Psi_{cl}$

$$\{0\} \to D \leftarrow D^* \leftarrow \widetilde{D}^*$$

universal cover of punctured disc D^* near 0

$$(X_0)_{cl} \xrightarrow{i} f_{cl}^{-1}(D) \xleftarrow{\bar{j}} f_{cl}^{-1}(\widetilde{D}^*)$$

$$R\Psi_{cl} : D^+((X - X_0)_{cl}, \Lambda) \to D^+((X_0)_{cl}, \Lambda)$$

$$R\Psi_{cl}(F) = i^* R\bar{j}_*(F|f_{cl}^{-1}(\widetilde{D}^*))$$

$+$ action of $\pi_1(D^*) = \mathrm{Aut}(\widetilde{D}^*/D) \xrightarrow{\sim} \mathbf{Z}$

$$R\Psi_{cl} : D^+((X - X_0)_{cl}, \Lambda) \to D^+((X_0)_{cl} \times B\pi_1(D^*), \Lambda)$$

- Comparison map

$$\varepsilon : (X_0)_{cl} \times B\pi_1(D^*) \to X_0 \times \eta$$

$$(*) \qquad \varepsilon^* R\Psi K \to R\Psi_{cl}(\varepsilon^* K)$$

(in $D((X_0)_{cl} \times B\mathbf{Z}, \Lambda)$)

To define ε, relate $\widetilde{D}^*$ and $\overline{\eta}$ as follows :

$k(\overline{\eta}) = \{$ germs at 0 of holomorphic functions on $\widetilde{D}^*$ algebraic over field of functions of Y $\}$

Define (*) by approximation, writing normalization of S in $\overline{\eta}$ as an inverse limit of affine Y-schemes of finite type, and using previous comparison map for finite type $\mathbf{C}$-schemes

details in SGA 7 X XIV

Theorem

For $K \in D_c^+(X - X_0, \Lambda)$

$$(*) \qquad \varepsilon^* R\Psi K \to R\Psi_{cl}(\varepsilon^* K)$$

is an isomorphism

In particular :

Corollary

$$(R\Psi_{cl}\mathbf{Z}) \otimes \mathbf{Z}_\ell \xrightarrow{\sim} R\Psi \mathbf{Z}_\ell$$

4. Examples

Even for $F = \Lambda$, $R\Psi F$ explicitly calculated in very few cases :

- Semistable reduction (and variants)
- Quadratic singularities

4.1. Semistable reduction

S : strictly local trait, $s \to S \leftarrow \eta$

X/S semistable reduction $\Leftrightarrow_{\mathrm{def}}$ X flat, ft/ S, X_η smooth, X regular, and $X_s \subset X$ = reduced divisor with normal crossings

$\Leftrightarrow$ étale locally on X, X isomorphic to $S[t_1, \cdots, t_n]/(t_1 \cdots t_r - \pi)$
(π = uniformizing parameter in R, $S = \mathrm{Spec}\, R$) ;
$X_s = V(t_1 ... t_r) \subset X$; $\dim X = n$)

strict semistable : $Y := X_s$ is a strict normal crossings divisor :
$Y = \sum_{1 \le i \le r} Y_i$, Y_i regular, irreducible

$\Lambda = \mathbf{Z}/\ell^\nu \mathbf{Z}$, ℓ invertible on S ; $R\Psi\Lambda$ given by following th :

Theorem

(1) $R\Psi\Lambda = R\Psi_{\mathrm{t}}\Lambda$ *(*$R\Psi\Lambda$ *tame)*
(2) $R^0\Psi\Lambda = \Lambda_Y$
(3) $0 \to \Lambda_Y \to \oplus_i \Lambda_{Y_i} \to R^1\Psi\Lambda \to 0$
(4) $\Lambda^q R^1\Psi\Lambda \xrightarrow{\sim} R^q\Psi\Lambda$
(5) $I = \mathrm{Gal}(\overline{\eta}/\eta)$ *acts trivially on* $R^q\Psi\Lambda$ *for all* q*, unipotently on* $R\Psi\Lambda$*.*

Remarks

- $R\Psi_{\mathrm{t}}\Lambda$ calculated by Grothendieck-Deligne (SGA 7 I) assuming Grothendieck's absolute purity conjecture for divisor $Y \subset X$

- tameness and full calculation by Rapoport-Zink (1982)

- general absolute purity conjecture proved by Gabber (1994), new proof in 2005

- generalization of theorem to log smooth case (Nakayama, 1998)

- simplified proof of tameness and purity conjecture (for $Y \subset X$) : (I., 2004)

(5) $I = \mathrm{Gal}(\overline{\eta}/\eta)$ acts unipotently on $R\Psi\Lambda$

$\Rightarrow$ existence of monodromy operator

$$N : R\Psi\Lambda \to R\Psi\Lambda(-1)$$

(in $D(Y, \Lambda)$), satisfying $N^{n+1} = 0$, characterized by

$$\sigma | R\Psi\Lambda = \exp(N t_\ell(\sigma) : R\Psi\Lambda \to R\Psi\Lambda)$$

for $\sigma \in I$, where $t_\ell : I \to \mathbf{Z}_\ell(1) = \ell$-component of tame character

• explicit description of N by Rapoport-Zink, using ℓ-adic variant of Steenbrink's double complex, and calculation of monodromy filtration

• Calculation of monodromy filtration and other filtrations associated with N, using perversity of $R\Psi\Lambda[n]$ (T. Saito, 2003), applications to weight spectral sequence

Sketch of proof of (1) : tameness of $R\Psi\Lambda$

$Y = X_s = \sum Y_i$ sncd in X ; for $x \to Y$ geometric pt, define $r(x)$ = number of branches of Y through x,

$$r(X) = \sup_{x \to Y} r(x)$$

$(1 \leq r(X) < +\infty)$

Proof of tameness of $R\Psi\Lambda$ by induction on $r(X)$. Assume tameness holds for $r(X) \leq r$ (reduction with at most r branches), wants to prove it for $r(X) = r + 1$.
WMA $X = S[t_1, \cdots, t_n]/(t_1 \cdots t_{r+1} - \pi)$, then (functoriality for smooth maps) WMA

$$X = S[t_1, \cdots, t_{r+1}]/(t_1 \cdots t_{r+1} - \pi).$$

Let

$$0 = V(t_1, \cdots, t_{r+1}) \in Y$$

Induction assumption $\Rightarrow R\Psi\Lambda|Y - \{0\}$ tame. Want to show $(R\Psi\Lambda)_0$ tame.

Define wild quotient $R\Psi_w\Lambda$ by exact triangle

$$R\Psi_t\Lambda \to R\Psi\Lambda \to R\Psi_w\Lambda \to$$

Then

$$R\Psi_w\Lambda = (R\Psi_w\Lambda)_0$$

and want to show $(R\Psi_w\Lambda)_0 = 0$.

Key observation : semistable reduction with n branches can be obtained from smooth map by successive blow up of smooth divisors in special fiber

Let

$$Z := S[t_1, \cdots, t_{r+1}]/(t_1 \cdots t_r - \pi),$$

$$C := V(t_r, t_{r+1}) \subset Z,$$

and

$$Z' := \mathrm{Bl}_C(Z) \xrightarrow{f} Z \to S$$

Then $r(Z) = r$, while $r(Z'/S) = r + 1$

more precisely, if $E =$ exceptional divisor, and $x_0 \in E =$ intersection of strict transforms of $t_i = 0$ for $i \leq r$, then

$$r_{Z'}(x) \begin{cases} \leq r & \text{if } x \neq x_0 \\ = r + 1 & \text{if } x = x_0 \end{cases}$$

$\Rightarrow$ may replace $(X, 0)$ by (Z', x_0)

Use functoriality of $R\Psi$ for proper push forward by $f : Z' \to Z$, get

$$(R\Psi_{X,w}\Lambda)_0 = (R\Psi_{Z',w}\Lambda)_{x_0} = (Rf_*(R\Psi_{Z',w}\Lambda))_{f(x_0)} = (R\Psi_{Z,w}\Lambda)_{f(x_0)} = 0$$

by induction assumption

Review of absolute purity theorem

Let

$$i : Y \to X$$

closed immersion of everywhere codimension d, X, Y regular ; $\Lambda = \mathbf{Z}/n\mathbf{Z}$, n invertible on X. Then : Grothendieck's absolute purity conjecture is Gabber's theorem :

Theorem

$$Ri^! \Lambda_X = \Lambda_Y[-2d](-d)$$

i. e.

$$\mathcal{H}^q_Y(\Lambda) = \begin{cases} 0 & \text{if } q \neq 2d \\ \Lambda(-d) & \text{if } q = 2d \end{cases}$$

with (for Y connected)

$$\Lambda \xrightarrow{\sim} H^0(Y, \mathcal{H}^{2d}_Y(\Lambda))(d) = H^{2d}_Y(X, \Lambda(d))$$

given by cohomology class of Y.

Mostly used through

Corollary

$D = \sum_{1\le i\le r}$ sncd in X, $j : U = X - D \to X$, then

$$R^q j_*\Lambda = \begin{cases} \Lambda & \text{if } q = 0 \\ \oplus_{1\le i\le r} \Lambda_{D_i}(-1) & \text{if } q = 1 \\ \Lambda^q R^1 j_*\Lambda & \text{if } q \ge 1 \end{cases}$$

with maps $\Lambda_{D_i}(-1) \to R^1 j_*\Lambda$ given by cohomology class of D_i

Sketch of proof of (2) - (5)

Calculation on stalks. Replace X by strict localization at $x \in X_s$.
WMA : $X = S\{t_1, \cdots, t_r\}/(t_1 \cdots t_r - \pi)$.

tameness $\Rightarrow$

$$(R^q\Psi\Lambda)_x = H^q(X_{\eta_t}, \Lambda)$$

where $\eta_t = \varprojlim_{(n,p)=1} \eta(\pi^{1/n})$ (maximal tame extension of η)
Let $U = X_\eta = X - Y$, $Y = X_s = V(t_1 \cdots t_r)$, and

$$\widetilde{U} = \varprojlim_{(n,p)=1} U[t_1^{1/n}, \cdots, t_r^{1/n}] \to U$$

the tame universal cover of U

Let $Z := \widehat{\mathbf{Z}}'(1) = \varprojlim_{(n,p)=1} \mu_n(k)$. Have fibrations

$$(*) \qquad \begin{tikzcd} & \widetilde{U} \arrow[dl, "Z^r"'] \arrow[d, "Z^{r-1}"] \\ U & X_{\eta_t} \arrow[l, "Z"'] \end{tikzcd} .$$

Absolute purity $\Rightarrow$ (*) cohomologically of the form

$$1 \to BZ^{r-1} \to BZ^r \to BZ \to 1$$

corresponding to split exact sequence

$$(**) \qquad 0 \to Z^{r-1} \to Z^r \overset{(m_1,\cdots,m_r)\mapsto\sum m_i}{\to} Z \to 0.$$

Main point :

$$H^q(\widetilde{U}, \Lambda) = \begin{cases} \Lambda & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases}$$

(absolute purity $\Rightarrow$ for $q > 0$ transition map of inductive system $H^q(U[t_1^{1/n}, \cdots, t_r^{1/n}], \Lambda)$ are essentially zero)

Then :

$$H^q(X_{\eta_t}, \Lambda) = H^q(Z^{r-1}, \Lambda) = \Lambda^q H^1(Z^{r-1}, \Lambda),$$

$$0 \to \Lambda \to \Lambda^r \to H^1(X_{\eta_t}, \Lambda) \to 0$$

$$(**) \qquad 0 \to Z^{r-1} \to Z^r \overset{(m_1, \cdots, m_r) \mapsto \sum m_i}{\to} Z \to 0.$$

split $\Rightarrow$ Z $(= I_t)$ acts trivially on $R^q\Psi\Lambda$ (X_{η_t} connected)

Bound on the unipotence exponent

Corollary

X/S semistable reduction, proper ; $r(X)$ maximum number of branches of $Y = X_s$ through a point. For $\sigma \in I$,

$$(\sigma - 1)^N | H^q(X_{\overline{\eta}}, \Lambda) = 0$$

for $N \geq \inf(q+1, r(X))$.

Proof : Use :

- $R^q\Psi\Lambda = 0$ for $q \geq r(X)$
- $\sigma - 1 = 0$ on $R^q\Psi\Lambda$
- $H^q(X_{\overline{\eta}}, \Lambda) = H^q(X_s, R\Psi\Lambda)$.

Variants with higher multiplicities

Th. generalized by Nakayama (1998) to log smooth map $f : X \to S$ between fs log schemes. In particular, for X/S with generalized semistable reduction, i. e. étale loc. of the form

$$X = S[t_1, \cdots, t_n]/(t_1^{a_1} \cdots t_r^{a_r} - \pi)$$

with $\gcd(p, a_1, \cdots, a_r) = 1$. Again, $R\Psi\Lambda$ tame. However, I no longer acts trivially on $R^q\Psi\Lambda$. In strictly local case, X_{η_t} no longer connected,

$$\pi_0 = \pi_0(X_{\eta_t}) = \mathrm{Coker}(Z^r \to Z), \ \ (m_i) \mapsto \sum a_i m_i,$$

transitively permuted by $Z(= I_t)$, and

$$R^q\Psi\Lambda = \Lambda[\pi_0] \otimes \Lambda^q H^1(Z^{r-1}, \Lambda),$$

with action of I through π_0 (regular representation). See also I.'s Overview in Astérisque 279.

4.2. Isolated singularities

Theorem

S = strictly local trait ; X regular, flat, finite type over S, rel. dim n, smooth outside closed point $x \in X_s$. Then $R\Phi\Lambda | X_s - \{x\} = 0$ and

$$(R\Phi^q\Lambda)_x = \begin{cases} 0 & \text{if } q \neq n \\ \Lambda^r & \text{if } q = n \end{cases}$$

Remark Assume $k = k(s)$ alg. closed. If $\mathrm{char}(k) = 0$, or (more generallly) $R\Psi\Lambda$ tame (i. e. $R^n\Phi\Lambda$ tame), then

$$r = \mu = \mu(X/S, x)$$

Milnor number of X/S at x, $= \dim T^1_{X/S}(x)$, e. g. for X/S deduced from $f : Z = \mathbf{A}_k^{n+1} \to \mathbf{A}_k^1$ by localization, $x = 0$, $f(0) = 0$, then

$$\mu = \dim_k \mathcal{O}_{Z,x}/(\partial f/\partial x_0, \cdots, \partial f/\partial x_n).$$

In general :

Deligne-Milnor conjecture

$$\mu = r + \mathrm{sw}(R^n\Phi\Lambda),$$

$\mathrm{sw}(R^n\Phi\Lambda)$ = Swan conductor, measuring wild ramification, $= 0$ in tame case

proved by Deligne (SGA 7 XVI) if S of equal characteristic. Mixed char. case still open.

4.3. Quadratic singularities (SGA 7 XV)

Assume k alg. closed.

Theorem

In previous th. assume x = ordinary quadratic singularity. Then $r = 1$, i. e.

$$(R^n\Phi\Lambda)_x = \Lambda$$

ordinary quadratic singularity means :

- $n = 2m - 1$: X étale loc. near x isom. to

$$V(\sum_{1 \leq i \leq m} x_i x_{i+m} + \pi) \subset \mathbf{A}_S^{2m}$$

near $\{0\}$ (π = uniformizing parameter)

- $n = 2m$: X étale loc. near x isom. to

$$\begin{cases} V(\sum_{1 \leq i \leq m} x_i x_{i+m} + x_{2m+1}^2 + \pi) \subset \mathbf{A}_S^{2m+1} & \text{if } p > 2 \\ V(\sum_{1 \leq i \leq m} x_i x_{i+m} + x_{2m+1}^2 + a x_{2m+1} + \pi) \subset \mathbf{A}_S^{2m+1} & \text{if } p = 2 \end{cases}$$

near $\{0\}$ ($a \in \mathfrak{m}$, $a^2 - 4\pi \neq 0$).

Action of inertia I on $R^n\Phi\Lambda$:

- trivial if n odd
- through character ε of order 2 if n even, tame if $p > 2$.

For X/S proper, flat, rel. dim. n, having isolated singularities, i. e. smooth outside finite $\Sigma \subset X_s$,

$$R\Phi\Lambda = \oplus_{x\in\Sigma}(R\Phi\Lambda)_x$$

Specialization sequence for $K = \Lambda$

$$\cdots \to H^i(X_{\bar{s}}, K|X_{\bar{s}}) \xrightarrow{\mathrm{sp}} H^i(X_{\bar{\eta}}, K|X_{\bar{\eta}}) \to H^i(X_{\bar{s}}, R\Phi K) \to \cdots$$

boils down to interesting part

$$0 \to H^n(X_{\bar{s}}, \Lambda) \xrightarrow{\mathrm{sp}} H^n(X_{\bar{\eta}}, \Lambda) \xrightarrow{\varphi} \oplus_x (R^n\Phi\Lambda)_x \to$$

$$H^{n+1}(X_s, \Lambda) \to H^{n+1}(X_{\bar{\eta}}, \Lambda) \to 0.$$

• In isolated quadratic singularity case (and X smooth outside x), knowledge of $(R^n\Phi\Lambda)_x \xrightarrow{\sim} \Lambda$ (non canonical) doesn't suffice to calculate

$$\varphi : H^n(X_{\overline{\eta}}, \Lambda) \to (R^n\Phi\Lambda)_x.$$

Needs duality between $(R^n\Phi\Lambda)_x$ and $H^n_{\{x\}}(X_s, R\Psi\Lambda)$, i. e. perfect pairing

$$\langle , \rangle : H^n_{\{x\}}(X_s, R\Psi\Lambda) \otimes (R^n\Phi\Lambda)_x \to \Lambda$$

and identification of a distinguished generator δ_x of $H^n_{\{x\}}(X_s, R\Psi\Lambda)$ defined up to sign, called the vanishing cycle at x, so that φ given by

$$\langle \delta_x, \varphi a \rangle = \mathrm{Tr}(\widetilde{\delta}_x . a)$$

($\widetilde{\delta}_x$ = image of δ_x in $H^n(X_s, \Lambda)$, $\mathrm{Tr} : H^{2n}(X_s, \Lambda) \to \Lambda$ = trace map, Tate twists ignored)

• Knowledge of action of I on $R^q\Phi\Lambda$ (or $R^q\Psi\Lambda$) does,n't suffice to determine action of I on $H^n(X_{\overline{\eta}}, \Lambda)$. For $\sigma \in I$, needs variation

$$\mathrm{Var}(\sigma) : (R^n\Phi\Lambda)_x \to H^n_{\{x\}}(X_s, R\Psi\Lambda)$$

factoring $\sigma - 1$:

$$\begin{tikzcd} H^n(X_s, \Lambda) \arrow[r] \arrow[d, "\sigma-1"'] & (R^n\Phi\Lambda)_x \arrow[d, "\mathrm{Var}(\sigma)"'] \\ H^n(X_s, \Lambda) & H^n_x(X_s, \Lambda) \arrow[l] \end{tikzcd}$$

For quadratic singularities, $\mathrm{Var}(\sigma)$ given by Picard-Lefschetz formula

$$\mathrm{Var}(\sigma)a = \begin{cases} (-1)^{m\frac{\varepsilon_x(\sigma)-1}{2}}\langle\delta_x, a\rangle\delta_x & \text{if } n = 2m \\ (-1)^{m+1}t_\ell(\sigma)\langle\delta_x, a\rangle\delta_x & \text{if } n = 2m-1 \end{cases}$$

$\varepsilon_x : I \to \pm 1$ tame if $p > 2$, defined by $t^2 + at + \pi = 0$, if $p = 2$ and local form of X near x is

$$V(\sum_{1\leq i\leq m} x_i x_{i+m} + x_{2m+1}^2 + a x_{2m+1} + \pi).$$

Proof of PL : • SGA 7 XV : by transcendental argument and comparison th. for n odd

• alg. proof : I. (2000), by reduction to semistable reduction with 2 branches.

PL : • key point in Grothendieck's semistable reduction theorem for abelian varieties

• starting point of cohomological theory of Lefschetz pencils ($\Rightarrow$ Weil I, II)

5. Grothendieck's local monodromy theorem

Here $\Lambda = \mathbf{Q}_\ell$.

Theorem

$s \to S \leftarrow \eta$: henselian trait, $k = k(s)$, $p = char(k)$, $\ell \neq p$;
$I \subset \mathrm{Gal}(\overline{\eta}/\eta)$: the inertia group
X/S separated, finite type ; $i \in \mathbf{Z}$;

$$H^i := \begin{cases} H^i(X_{\overline{\eta}}, \mathbf{Q}_\ell) \\ \text{or } H^i_c(X_{\overline{\eta}}, \mathbf{Q}_\ell) \end{cases}$$

Then there exists an open subgroup $I_1 \subset I$, independent of ℓ, such that

$$\sigma \in I_1 \Rightarrow \sigma | H^i \text{ unipotent}$$

History of the theorem

• Grothendieck (1967) gave 2 proofs of th. (without the complement on independence on ℓ, and only one being unconditional) :

(1) arithmetic proof for $H^i = H^i_c(X_{\overline{\eta}}, \mathbf{Q}_\ell)$ (finiteness of $H^i(X_{\overline{\eta}}, \mathbf{Q}_\ell)$ unknown at the time): unconditional, relying on Grothendieck's local monodromy lemma

(2) geometric proof for $p = 0$, using resolution of singularities, absolute purity (available thanks to Artin), and calculation of $R^q\Psi\Lambda$ in generalized semistable reduction case (and $p = 0$)

Grothendieck deduced from (2) : Milnor's quasi-unipotence conjecture for monodromy of isolated singularities ($/\mathbf{C}$)

• Deligne (1996), using de Jong's alterations, made proof (2) work unconditionally, with complement on independence of ℓ (Berthelot's Bourbaki exposé 815)

Sketch of arithmetic proof

• special case : k finitely generated (or radicial over field finitely generated) over prime field
Then : H^i = continuous, finite dimensional representation of $G_K = \mathrm{Gal}(\overline{\eta}/\eta)$. Apply Grothendieck's local monodromy lemma

• general case : reduce to special case by spreading out, using Néron's desingularization, and generic constructibility for $R^i f_*$ or $R^i f_!$ (SGA 7 I 1.3)

Sketch of geometric proof, using de Jong

• WMA S complete : if $K = k(\eta)$, $\mathrm{Gal}(\overline{\widehat{K}}/\widehat{K}) \xrightarrow{\sim} \mathrm{Gal}(\overline{K}/K)$ (SGA 4 X 2.2.1) ($S = \mathrm{Spec}(R)$, $\widehat{K} := \mathrm{Frac}(\widehat{R})$)

• Th. OK if X/S proper, semistable : $(\sigma - 1)^{i+1}|H^i = 0$

• Th. OK if X_η proper, smooth. Choose finite extension η_1/η s. t. components of X_{η_1} are geometrically connected, replace X_η by component Z of X_{η_1}, then apply de Jong's theorem (possible as S complete) :

There exists : finite extension η_2 of η_1,
alteration $a : Z_2 \to Z$ over η_2
proper semistable model X_2/S_2 of Z_2, $S_2 =$ normalization of S in η_2.
composition $h : Z_2 \xrightarrow{a} Z \to X_\eta$ proper, generically finite, degree d $\Rightarrow$

$$\mathbf{Q}_\ell \to Rh_*\mathbf{Q}_\ell \xrightarrow{\mathrm{Tr}} \mathbf{Q}_\ell$$

is multiplication by d ; $\Rightarrow H^i(X_{\overline{\eta}}, \mathbf{Q}_\ell) \hookrightarrow H^i((X_2)_{\overline{\eta}}, \mathbf{Q}_\ell)$, OK by first case (proper, semistable)

- general case for H^i_c : use induction on $\dim(X_{\eta)}$, and de Jong (over fields) to reduce to previous case

- general case for H^i : use de Jong (over fields) and cohomological descent to reduce to X_η smooth, separated, then apply Poincaré duality between H^i and H^{2d-i}_c ($d = \dim(X_\eta)$)

6. The ℓ-adic weight spectral sequence

6.1. Direct proof of perversity of $R\Psi\Lambda[n]$ in semistable case

$s \to S \leftarrow \eta$ strictly local trait, X/S strict semistable reduction, $Y = X_s = \sum_{1 \leq i \leq r} Y_i$ sncd, $\Lambda = \mathbf{Z}/\ell^\nu\mathbf{Z}$, $\dim(Y) = n$ as in 4.1

$$Y = X_s \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} X_\eta$$

$R\Psi\Lambda$ tame $\Rightarrow$

$$i^* Rj_*\Lambda = R\Gamma(I_t, R\Psi\Lambda)$$

$$(*) \qquad E_2^{ij} = H^i(I_t, R^j\Psi\Lambda) \Rightarrow H^{i+j}(I_t, R^j\Psi\Lambda) = i^* R^{i+j} j_*\Lambda$$

concentrated on columns $i = 0$, $i = 1$ as $I_t = \widehat{\mathbf{Z}}'(1)$

trivial action of I on $R^j\Psi\Lambda \Rightarrow$

$H^0(I_t, R^j\Psi\Lambda) = H^1(I_t, R^j\Psi\Lambda(1)) = R^j\Psi\Lambda$, hence

$$(*) \qquad E_2^{ij} = H^i(I_t, R^j\Psi\Lambda) \Rightarrow H^{i+j}(I_t, R^j\Psi\Lambda) = i^*R_*^{i+j}j_*\Lambda$$

gives short exact sequences

$$0 \to R^q\Psi\Lambda(q) \to i^*R^{q+1}j_*\Lambda(q+1) \to R^{q+1}\Psi\Lambda(q+1) \to 0,$$

spliced together into a resolution

$$(**) \qquad 0 \to \Lambda_Y \xrightarrow{\theta} i^*R^1j_*\Lambda(1) \xrightarrow{\theta} \cdots \xrightarrow{\theta} i^*R^{n+1}j_*\Lambda(n+1) \to 0,$$

with $\theta =$ cup product with tautological class in $H^1(I_t, \Lambda(1))$

$$(**) \qquad 0 \to \Lambda_Y \xrightarrow{\theta} i^* R^1 j_* \Lambda(1) \xrightarrow{\theta} \cdots \xrightarrow{\theta} i^* R^{n+1} j_* \Lambda(n+1) \to 0,$$

absolute purity $\Rightarrow$ (**) isomorphic to

$$(***) \qquad 0 \to \Lambda_Y \to a_{0*}\Lambda \xrightarrow{d} \cdots \xrightarrow{d} a_{n*}\Lambda \to 0,$$

where $Y_J = \cap_{j \in J} Y_j$

$$a_m : Y^{(m)} := \coprod_{|J|=m+1} \to Y_J,$$

d = Cech differential. In particular, get resolution

$$(****) \qquad 0 \to R^q\Psi\Lambda(q) \to a_{q*}\Lambda \to \cdots \to a_{n*}\Lambda \to 0$$

$\dim(Y^{(m)}) = n - m \Rightarrow \Lambda[n-m]$ perverse on $Y^{(m)} \Rightarrow a_{m*}\Lambda[n-m]$ perverse on $Y \Rightarrow R^q\Psi\Lambda[n-q]$ perverse on Y

$\Rightarrow R\Psi\Lambda[n]$ perverse on Y, as predicted by Gabber's theorem.

6.2. Monodromy, kernel, and image filtrations

X/S as in 6.1, but $\Lambda = \mathbf{Q}_\ell$. Recall monodromy operator

$$N : R\Psi\Lambda \to R\Psi\Lambda(-1)$$

(in $D(Y,\Lambda)$), satisfying $N^{n+1} = 0$ ($n = \dim(Y)$), characterized by

$$\sigma | R\Psi\Lambda = \exp(Nt_\ell(\sigma) : R\Psi\Lambda \to R\Psi\Lambda)$$

for $\sigma \in I$, where $t_\ell : I \to \mathbf{Z}_\ell(1) =$ ℓ-component of tame character. As

$$R\Psi\Lambda \in \mathrm{Per}(Y)[-n],$$

N is a (Tate twisted) nilpotent endomorphism of $R\Psi\Lambda$ in the abelian category $\mathrm{Per}(Y)[-n]$

hence N defines 3 filtrations on $R\Psi\Lambda$:

- kernel filtration

$$F_i = \mathrm{Ker}\, N^{i+1},\ 0 = F_{-1} \subset F_0 \subset \cdots \subset F_n = R\Psi\Lambda,$$

- image filtration

$$G_j = \mathrm{Im}\, N^j,\ R\Psi\Lambda = G^0 \supset G^1 \supset \cdots \supset G^n \supset G^{n+1} = 0,$$

- monodromy filtration

$$M_r = \sum_{i-j=r} F_i \cap G^j,$$

characterized by

$$N(M_k) \subset M_{k-2}(-1)$$

and

$$N^k : gr_k^M R\Psi\Lambda \xrightarrow{\sim} gr_{-k}^M R\Psi\Lambda(-k).$$

Associated graded given by

$$gr_k^M R\Psi\Lambda = \bigoplus_{p-q=k} gr_p^F gr_G^q R\Psi\Lambda.$$

T. Saito (2003) explicitly determined :

- kernel filtration :

$$F_p = \tau_{\leq p} R\Psi\Lambda$$

(canonical truncation) ; in particular, $\mathrm{gr}_p^F = R^p\Psi\Lambda[-p]$

- trace on gr_p^F of image filtration : via the resolution

$$(****) \qquad 0 \to R^p\Psi\Lambda(p) \to a_{p*}\Lambda \to \cdots \to a_{n*}\Lambda \to 0,$$

$$G^q \mathrm{gr}_p^F = (0 \to a_{p+q*}\Lambda \to \cdots \to a_{n*}\Lambda \to 0)(-p)$$

(naïve filtration) (with $a_{n*}\Lambda$ in degree n)

- associated graded for monodromy filtration :

$$\mathrm{gr}_p^F \mathrm{gr}_G^q = (a_{p+q*}\Lambda)[-p-q](-p)$$

Method of proof : use description of N given by Rapoport-Zink bicomplex $A^{\bullet,\bullet}$

Definition of $A^{\bullet,\bullet}$: choose complex

$$K = (K^0 \to K^1 \cdots \to K^i \to \cdots)$$

of $\Lambda[\mathbf{Z}_\ell(1)]$-modules on Y representing $R\Psi\Lambda$. Choose topological generator T of $\mathbf{Z}_\ell(1)$. Then

$$i^* Rj_* \Lambda \xrightarrow{\sim} M := \mathrm{s}(K \xrightarrow{T-1} K)$$

(where s = associated simple complex). Define

$$\begin{tikzcd} K(1) \arrow[r, "1-T"] & K(1) & & M(1)[1] \\ & K \arrow[u, "1\otimes T"] \arrow[r, "T-1"] & K & M \arrow[u, "\theta"] \end{tikzcd}$$

$$L^q := (\tau_{\geq q+1} M)(q+1)[q+1]$$

$$A^{\bullet,\bullet} = \mathrm{s}(q \mapsto A^{\bullet,q} = L^q, \theta : L^q \to L^{q+1})$$

$A^{\bullet,\bullet}$ contained in first quadrant :

$$
\begin{tikzcd}
\vdots & \vdots \\
A^{\bullet,q} \arrow[u] & (\tau_{\geq q+1} M)(q+1)[q+1] \arrow[u, "\theta"] \\
\vdots & \vdots \\
A^{\bullet,1} \arrow[u] & (\tau_{\geq 2} M)(2)[2] \arrow[u, "\theta"] \\
A^{\bullet,0} \arrow[u] & (\tau_{\geq 1} M)(1)[1] \arrow[u, "\theta"]
\end{tikzcd}
$$

with augmentation

$$\varepsilon : K(= R\Psi\Lambda) \to A^{\bullet,\bullet}$$

induced by $1 \otimes T : K \to M(1)[1]$. Exact sequences

$$(****) \qquad 0 \to R^q\Psi\Lambda(q) \to a_{q*}\Lambda \to \cdots \to a_{n*}\Lambda \to 0$$

$\Rightarrow \varepsilon$ induces exact sequences on cohomology columns, hence an isomorphism (in $D^+(Y, \Lambda[\mathbf{Z}_\ell(1)])$

$$\varepsilon : R\Psi\Lambda \xrightarrow{\sim} \mathrm{s}A^{\bullet,\bullet}.$$

Advantage of $A^{\bullet,\bullet}$: (for $\Lambda = \mathbf{Q}_\ell$)

$N : R\Psi\Lambda \to R\Psi\Lambda(-1)$ becomes visible :

$$N = ((T-1) \otimes T^\vee).u,$$

u an automorphism. The nilpotent endomorphism

$$\widetilde{N} := (T-1) \otimes T^\vee : R\Psi\Lambda \to R\Psi\Lambda(-1)$$

($T^\vee \in \mathbf{Z}_\ell(-1)$ dual of T), which makes sense for $\Lambda = \mathbf{Z}/\ell^\nu\mathbf{Z}$, is induced from the endomorphism

$$\nu : A^{\bullet,\bullet} \to A^{\bullet-1,\bullet+1}(-1),$$

$\nu|A^{i,j} := (-1)^{i+j+1}$ canonical projection $A^{i,j} \to A^{i-1,j+1}(-1)$

and monodromy filtration $M_\bullet R\Psi\Lambda$ given by

$$M_r R\Psi\Lambda = \mathrm{s}W_r A^{\bullet,\bullet} := \mathrm{s}(q \mapsto \tau_{\leq r+q} A^{\bullet,q})$$

($\mathrm{s}W_\bullet$ sometimes called (shifted) weight filtration)

6.3. The weight spectral sequence

X/S proper, strictly semistable, $\Lambda = \mathbf{Q}_\ell$

Filtration M_r on $R\Psi\Lambda$ in $\mathrm{Per}(Y)[-n]$

$\mapsto$ quasi-filtration (or spectral object $M_{[p,q]}R\Psi\Lambda$) in $D_c^b(Y,\Lambda)$

$\mapsto$ spectral sequence

$$(*) \qquad E_1^{i,j} = H^{i+j}(Y, \mathrm{gr}_{-i}^M R\Psi\Lambda) \Rightarrow H^{i+j}(X_{\overline{\eta}}, \Lambda),$$

called weight spectral sequence.

Alternate definition : (*) = spectral sequence of filtered complex

$$(\mathsf{s}A^{\bullet,\bullet}, \mathsf{s}W_\bullet)$$

Recall

$$\mathrm{gr}_k^M R\Psi\Lambda = \bigoplus_{p-q=k} \mathrm{gr}_p^F \mathrm{gr}_G^q R\Psi\Lambda,$$

$$\mathrm{gr}_p^F \mathrm{gr}_G^q = (a_{p+q*}\Lambda)[-p-q](-p).$$

$\Rightarrow$ in total degree m

$$E_1^{-r,m+r} = \oplus_{q\geq 0, r+q\geq 0} H^{m-r-2q}(Y^{(r+1+2q)}, \mathbf{Q}_\ell)(-r-q)$$

differential d_1 = sum of restriction and Gysin maps ((E_1, d_1) depends only on Y)

(but (*) does depend on X, actually only on $X \otimes R/(\pi^2)$ (Nakayama))

Arithmetic case

Assume $S = S_{0(s)}$, strict localization of henselian trait $s_0 \to S_0 \leftarrow \eta_0$, and

$$(Y \xrightarrow{i} X \xleftarrow{j} X_\eta) = S \times_{S_0} (Y_0 \xrightarrow{i_0} X_0 \xleftarrow{j_0} X_{\eta_0})$$

with X_0/S_0 proper, strict semistable, rel. dim. n. Then $G := \mathrm{Gal}(\overline{\eta}/\eta_0)$ acts on $R\Psi\Lambda$, compatibly with action on Y

N is G-equivariant, and weight spectral sequence

$$(*) \qquad E_1^{i,j} = H^{i+j}(Y, \mathrm{gr}^M_{-i} R\Psi\Lambda) \Rightarrow H^{i+j}(X_{\overline{\eta}}, \Lambda),$$

is G-equivariant.

Note : G acts on E_1 through $G_0 := Gal(k/k_0)$.

6.4. Main results and conjectures

Theorem

The weight spectral sequence

(∗)

$$E_1^{-r,m+r} = \oplus_{q \geq 0, r+q \geq 0} H^{m-r-2q}(Y^{(r+1+2q)}, \mathbf{Q}_\ell)(-r-q) \Rightarrow H^m(X_{\overline{\eta}}, \mathbf{Q}_\ell)$$

degenerates at E_2.

Indications on proof

• $\mathrm{char}(k) = 0$: reduce to X/S coming by localization from proper map X'/S', S' = smooth curve $/\mathbf{C}$, X'/S' having strict semistable reduction at $s \in S'$. Use comparison theorem with $R\Psi_{cl}\mathbf{C}$, and Hodge theory :

$E_1^{-r,m+r}$: pure Hodge structure of weight $m - r - 2q + 2r + 2q = m + r$, hence

$E_s^{-r,m+r}$: pure Hodge structure of weight $m + r$, hence

$d_s : E_s^{-r,m+r} \to E_s^{-r+s,m+r-s+1}$ vanishes for $s \geq 2$

- $k_0 = \mathbf{F}_q$, $X/S = S \times_{S_0} (X_0/S_0)$ as in arithmetic case. Let

$$F_q \in \mathrm{Gal}(k/k_0), \quad a \mapsto a^{1/q}$$

be the geometric Frobenius, and

$$F \in \mathrm{Gal}(\overline{\eta}/\eta_0) \mapsto F_q$$

a lifting. Then F defines an automorphism F^* of $R\Psi\mathbf{Q}_\ell$, hence an automorphism F^* of the weight spectral sequence $(*)$.

Deligne's Weil II $\Rightarrow$:

for all $1 \leq s \leq \infty$, $E_s^{-r,m+r}$ is pure of weight $m+r$, i. e. eigenvalues of F^* are q-Weil numbers of weight $m+r$

(NB. as inertia I acts unipotently, eigenvalues of F^* don't depend on choice of lifting F of F_q)

$\Rightarrow d_s = 0$ for $s \geq 2$

• General case. Two (independent) proofs (by reduction to arithmetic case)

- Nakayama (2000), using log geometry
- Ito (2005), using spreading out and Néron's desingularization as in arithmetic proof of local monodromy theorem

The following is the so-called weight monodromy conjecture

Conjecture

Define $\widetilde{M}_\bullet$:= abutment filtration of (*). Then

$\widetilde{M}_\bullet | H^m$ = monodromy filtration $M_\bullet$ of nilpotent endomorphism N of $H^m(X_{\overline{\eta}}, \mathbf{Q}_\ell)$.

$(N : H^* \to H^*(-1)$ defined by $\sigma | H^* = \exp(t_\ell(\sigma) N : H^* \to H^*)$ for $\sigma \in I)$

Remark

Conjecture means :

$$N^r : \mathrm{gr}_r^{\widetilde{M}} H^m \xrightarrow{\sim} \mathrm{gr}_{-r}^{\widetilde{M}} H^m(-r)$$

By definition, $\mathrm{gr}_r^{\widetilde{M}} H^m = E_\infty^{-r,m+r}$, and by degeneration at E_2,

$$E_\infty^{-r,m+r} = E_2^{-r,m+r}.$$

Therefore, conjecture $\Leftrightarrow$

$$N^r : E_2^{-r,m+r} \xrightarrow{\sim} E_2^{r,m-r}(-r)$$

Recall :

$$N = \widetilde{N}.\text{automorphism}$$

$$\widetilde{N} := (T-1) \otimes T^{\vee} : R\Psi\Lambda \to R\Psi\Lambda(-1)$$

($T^{\vee} \in \mathbf{Z}_{\ell}(-1)$ dual of T), induced by

$$\nu : A^{\bullet,\bullet} \to A^{\bullet-1,\bullet+1}(-1),$$

$\nu | A^{i,j} := (-1)^{i+j+1}$ canonical projection $A^{i,j} \to A^{i-1,j+1}(-1)$

and

$$\mathrm{gr}_r^{W_\bullet} \mathsf{s}A^{\bullet,\bullet} = \oplus_{p-q=r}\, a_{p+q*}\Lambda(-p),$$

hence

$$N^r : E_1^{-r,m+r} \xrightarrow{\sim} E_1^{r,m-r}(-r).$$

Main difficulty : $N^r | E_2$ involves model X/S, not just special fibre Y

To explain the name weight monodromy conjecture, needs

Interlude : the weight filtration

$s \to S \leftarrow \eta$ strict localization of
$s_0 \to S_0 \leftarrow \eta_0$: henselian trait, with $k_0 = k(s_0) = \mathbf{F}_q$, $\ell \neq p$

V : finite dimensional $\mathbf{Q}_\ell$-representation of $G = \mathrm{Gal}(\overline{\eta}/\eta_0)$.

Recall : inertia $I = \mathrm{Gal}(\overline{\eta}/\eta)$ acts quasi-unipotently on V : open subgroup $I_1 \subset I$ acts unipotently (Grothendieck's monodromy lemma). Implies :

Observation (Deligne) : Let F', F'' be liftings of F_q in G, and $\{\lambda'_1, \cdots, \lambda'_N\}$, $\{\lambda''_1, \cdots, \lambda''_N\}$ their sets of eigenvalues (in $\overline{\mathbf{Q}}_\ell$). Then there exist $n \geq 1$ s. t.

$$\{{\lambda'_1}^n, \cdots, {\lambda'_N}^n\} = \{{\lambda''_1}^n, \cdots, {\lambda''_N}^n\}$$

Consider condition

(A) For a lifting F of F_q, any eigenvalue λ of F is a q-Weil number (of weight $w = w(\lambda) \in \mathbf{Z}$)

Observation $\Rightarrow$: does not depends on choice of F (as roots of unity = q-Weil integers of weight 0)

Deligne [Weil II 1.7.5] :

Lemma

Assume V satisfies (A). Let

$$W(\overline{\eta}/\eta) = \{g \in \mathrm{Gal}(\overline{\eta}/\eta) \mapsto F_q^n \in \mathrm{Gal}(\overline{s}/s), n \in \mathbf{Z}\}$$

be the Weil group. Then there exists a unique $W(\overline{\eta}/\eta)$-stable finite increasing filtration

$$W_\bullet V,$$

called the weight filtration, s. t. $\mathrm{gr}_n^{W_\bullet} V$ pure of weight n.

Arithmetic rephrasing of WMC

X_0/S_0 proper, strictly semistable, $X/S = S \times_{S_0} (X_0/S_0)$.

Weil conjectures $\Rightarrow$ all $E_s^{-r,m+r}$ in weight monodromy spectral sequence satisfy (A). Moreover :

$\widetilde{M}_\bullet H^m$ $\mathrm{Gal}(\overline{\eta}/\eta)$-stable, and $\mathrm{gr}_r^{\widetilde{M}_\bullet} H^m$ pure of weight $m + r$ $(H^m = H^m(X_{\overline{\eta}}, \mathbf{Q}_\ell))$.

$\Rightarrow \widetilde{M}_{\bullet - m} H^m =$ weight filtration on H^m, i. e. $\widetilde{M}_r = W_{m+r}$

Hence : WMC $\Leftrightarrow$ (weight filtration) = (shifted monodromy filtration), i. e. $W_\bullet H^m = M_{\bullet - m} H^m$

Using de Jong's alterations, get :

Corollary

Let Z_0/η_0 *proper and smooth,* $Z = \eta \times_{\eta_0} Z_0$, $m \in \mathbf{Z}$. *Then :*

(a) $H^m = H^m(Z_{\overline{\eta}}, \mathbf{Q}_\ell)$ *satisfies (A).*

(b) Assume WMC holds. Then :

$M_{\bullet - m}H^m$ *= weight filtration on* H^m, *i. e.* $M_r H^m = W_{r+m}H^m$

where $M_\bullet$ *= monodromy filtration of nilpotent operator*
$N : H^m \to H^m(-1)$, $\sigma = \exp(t_\ell(\sigma)N)$ *for* $\sigma \in$ *suitable open* $I_1 \subset I$.

History and status of WMC

- WMC first appears in Deligne's Hodge I, §9, in the context of Hodge theory, for projective smooth varieties over an open disc, as a statement without proof. No proof given in Hodge II, III.
- same context : proof given by Steenbrink (1975) for semistable reduction case, but proof had a gap, found by ElZein
- proof corrected independently by Deligne (unpublished) and M. Saito in ([Modules de Hodge polarisables, RIMS 24, 1988], 4.2)
- arithmetic case (k_0 finite), equal characteristic, WMC (in the form of corollary) proved by Deligne (Weil II, 1.8.4)

- arithmetic case (k_0 finite), mixed characteristic, WMC proved by Rapoport-Zink (1982) for $\dim(X_\eta) \leq 2$
- general equicharacteristic case : WMC proved by Ito (2005)
- WMC proved for certain 3-folds X_η, or certain p-adically uniformized varieties X_η : Ito (2004, 2005)
- WMC proved for X_η set-theoretic complete intersection in projective space (or smooth toric projective variety) : Scholze (2011), using perfectoid spaces to reduce to equicharacteristic case

The local invariant cycle theorem

Notation and hypotheses of WMC.

Recall : nilpotent operator $N : R\Psi\mathbf{Q}_\ell \to R\Psi\mathbf{Q}_\ell(-1)$ defines kernel filtration

$$F_i = \mathrm{Ker}\, N^{i+1}, 0 = F_{-1} \subset F_0 \subset \cdots \subset F_n = R\Psi\mathbf{Q}_\ell,$$

hence $\mapsto$ quasi-filtration (or spectral object $F_{[p,q]}R\Psi\mathbf{Q}_\ell$) in $D^b_c(Y, \mathbf{Q}_\ell)$

$\mapsto$ spectral sequence

$$(K1) \qquad E_1^{i,j} = H^{i+j}(Y, \mathrm{gr}^F_{-i} R\Psi\mathbf{Q}_\ell) \Rightarrow H^{i+j}(X_{\overline{\eta}}, \mathbf{Q}_\ell),$$

Recall :

$$F_i R\Psi\mathbf{Q}_\ell = \tau_{\leq i} R\Psi\mathbf{Q}_\ell$$

$\Rightarrow$ up to renumbering,

$$(K1) \qquad E_1^{i,j} = H^{i+j}(Y, \mathrm{gr}^F_{-i} R\Psi\mathbf{Q}_\ell) \Rightarrow H^{i+j}(X_{\overline{\eta}}, \mathbf{Q}_\ell),$$

= 2nd spectral sequence of hypercohomology of Y with value in $R\Psi\mathbf{Q}_\ell$

$$(K2) \quad E_2^{i,j} = H^i(Y, R^j\Psi\mathbf{Q}_\ell) \Rightarrow H^{i+j}(Y, R\Psi\mathbf{Q}_\ell) = H^{i+j}(X_{\overline{\eta}}, \mathbf{Q}_\ell)$$

called spectral sequence of vanishing cycles

Corollary

Assume X satisfies WMC. Then (K1) (resp. (K2)) degenerates at E_2 (resp. E_3) and the abutment filtration is the kernel filtration : for (K2) we have

$$F^{m-r}H^m = \mathrm{Ker}\, N^{r+1} : H^m \to H^m(-r-1)$$

Proof : (almost) formal from WMC (M. Saito-Zucker) : use degeneration at E_2 of spectral sequence associated with filtration of $\mathrm{gr}_F R\Psi\mathbf{Q}_\ell$ cut-out by image filtration

As $H^m(X_{\overline{\eta}}, \mathbf{Q}_\ell)^I = \mathrm{Ker}(N : H^m \to H^m)$, get :

Corollary

Assume X satisfies WMC. Then :

(*lic*) $\quad \mathrm{Im}(\mathrm{sp} : H^m(X_{\overline{s}}, \mathbf{Q}_\ell \to H^m(X_{\overline{\eta}}, \mathbf{Q}_\ell)) = H^m(X_{\overline{\eta}}, \mathbf{Q}_\ell)^I$

Formula (lic) is called local invariant cycle theorem.

Remark Independently of WMC, in the equal char. case, Deligne (Weil II, 3.6.1) proves the more general (lic) :

Theorem

S = strict localization at a closed point of smooth curve over an alg. closed field k, X/S proper, s. t. X essentially smooth over k and $X_{\overline{\eta}}/\overline{\eta}$ smooth. Then (lic) holds, i. e.

$$\mathrm{Im}(\mathrm{sp} : H^m(X_s, \mathbf{Q}_\ell) \to H^m(X_{\overline{\eta}}, \mathbf{Q}_\ell)) = H^m(X_{\overline{\eta}}, \mathbf{Q}_\ell)^I$$

7. Further developments

- Hodge theory of nearby cycles (Steenbrink, M. Saito, ...)
- log nearby cycles (Kato, Nakayama, ...)
- ramification, characteristic cycles, Euler-Poincaré formulas, ℓ-adic Riemann-Roch (Deligne ; Laumon ; Abbes, T. Saito, Kato, ...)
- oriented products and nearby cycles over general bases (Deligne ; Sabbah ; Orgogozo, Gabber, ...)