The Category of M-sets

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Abstract

A topos is a category which looks and behaves very much like the category of sets, and so it may be thought of as a universe for mathematical discourses. One of the very useful topoi in many branches of mathematics as well as in computer sciences is the topos **MSet**, of sets with an action of a monoid on them. It is well known that **MSet**, being isomorphic to the functor category $\mathbf{Set}^M$, is a topos. Here, we explicitly give the ingredients of a topos in **MSet** and investigate their properties for the working scientists and computer scientists. Among other things, we give some equivalent conditions, such as the left Ore condition, to Ω, the subobject classifier of **MSet**, being a Stone algebra. Also the free and the cofree objects, as well as, limits and colimits are discussed in **MSet**.

1 MSet is a topos

In this section, as the referee suggested, we first briefly introduce the notion of a "topos".

The study of topoi arises within category theory. A category may be thought of in the first instance as a universe for a particular kind of mathematical discourses. Such a universe is determined by specifying a certain kind of "objects" and a certain kind of "arrows" that links different objects. The most general universe of current mathematical discourse is the category **Set** of sets with functions between them. Many basic properties of sets and set theoretic operations can be described by reference to the arrows in **Set**, and these descriptions can be interpreted in any category by means of its arrows. So the question that arises is "when does a category look and behave like **Set** ?" A vague answer is "when it is (at least) a topos". The word "topos" ("place" or "site" in Greek) was originally used by Alexander Grothendieck in a context of algebraic geometry. So, a topos is informally a category which looks and behaves very much like the category of sets.

A **topos** is formally a category which has finite limits, exponentiations (abstracting the fuction set B^A) and subobject classifier (abstracting the truth set $\mathbf{2} = \{0, 1\}$). Recall that, for a category $\mathcal{C}$ with finite products, we say that

$\mathcal{C}$ has *exponentiations* (exponentials) if for every objects A and B, there is an object B^A together with an arrow $ev : B^A \times A \to B$ (called *evaluation*) such that for every arrow $g : C \times A \to B$ there is a unique arrow $\hat{g} : C \to B^A$ with $ev \circ (\hat{g} \times id_A) = g$.

We also say that $\mathcal{C}$ has *subobject classifier* if there exists an object Ω with an arrow $t : \mathbf{1} \to \Omega$ (called *the truth arrow*) such that for every monomorphism $f : B \to A$ there is a unique arrow $\chi_f : A \to \Omega$ (called *the classifing arrow*) making the square

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ ! \downarrow & & \downarrow \chi_f \\ \mathbf{1} & \xrightarrow{t} & \Omega \end{array}$$

commutative.

Now, we formally introduce the category **MSet** and recall the proof of the fact that it is actually a topos.

Recall that, for a monoid M with e as its identity, a (left) M-set is a set X together with a function $\lambda : M \times X \to X$, called the action of M (or the M-action) on X, such that for $x \in X$ and $m, n \in M$ (denoting $\lambda(m, x)$ by mx)

i) $ex = x$

ii) $(mn)x = m(nx)$.

In fact, an M**-set** is an algebra $(X, (\lambda_m)_{m\in M})$ where each $\lambda_m : X \to X$ is a unary operation on X such that $\lambda_e = id_X$, $\lambda_m \circ \lambda_n = \lambda_{mn}$ for each $m, n \in M$.

A morphism $f : X \to Y$ between M-sets X, Y is an equivariant map; i.e. for $x \in X$, $m \in M$,

$$f(mx) = mf(x).$$

Since id_X and the composite of two equivariant maps are equivariant, we have the category **MSet** of all M-sets and equivariant maps between them.

As a very interesting example, used in computer sciences as a convenient mean of algebraic specification of process algebras (see [7, 8, 10]), consider the monoid $(\mathbb{N}^\infty, \cdot, \infty)$, where $\mathbb{N}$ is the set of natural numbers and $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$ with $n < \infty, \forall n \in \mathbb{N}$ and $m \cdot n = \min\{m, n\}$ for $m, n \in N^\infty$. Then an N^∞-set is called a projection algebra (see [7]).

1.1 Lemma *Considering the monoid M as a category M with one object, the category* **MSet** *is isomorphic to the functor category* $\mathbf{Set}^M$.

Proof: Define $\Phi : \mathbf{Set}^M \to \mathbf{MSet}$ as follows. Let, for $F : M \to \mathbf{Set}$, $\Phi(F) = FM$ with the action $mx = F(m)(x)$, for $m \in M$, $x \in FM$. For any natural transformation $\tau : F \to G$ in **MSet**, define $\Phi(\tau)$ to be the only component τ_M of τ. By the naturality of τ, one can see that τ_M is equivariant.

Conversely, define $\Psi : \mathbf{MSet} \to \mathbf{Set}^M$ as follows. For an M-set X, define $\Psi(X) : M \to \mathbf{Set}$ to be the functor given by $M \rightsquigarrow X$ and $(m : M \to M) \rightsquigarrow (\Psi(m) : X \to X)$ with $\Psi(m)(x) = mx$. Also, for each equivariant map $f :$

$X \to Y$, let $\Psi(f) : \Psi(X) \to \Psi(Y)$ be the natural transformation whose only component is f.

Now, one can easily check that Φ and Ψ are functors and $\Phi\Psi$=id, $\Psi\Phi$=id. This proves the lemma.$\square$

For a general case of the above lemma see 0.2.8. in [3].

1.2 Proposition *For any monoid M, the category* **MSet** *is a topos.*

proof: Since, for any (small) category $\mathcal{C}$, the functor category $\mathbf{Set}^{\mathcal{C}}$ is a topos, the proposition is just a corollary to the above lemma.$\square$

2 Free and cofree objects in MSet

Here, we construct a left and a right adjoint to the forgetful functor $U : \mathbf{MSet} \to \mathbf{Set}$. For the general case of this see [4].

2.1 Lemma *The forgetful functor U :* **MSet** $\to$ **Set** *has a left adjoint F :* **Set** $\to$ **MSet**.

Proof: Define $F : \mathbf{Set} \to \mathbf{MSet}$ by $F(X) = M \times X$ with the action given by $s(m, x) = (sm, x)$, for $s, m \in M$, $x \in X$, and for any map $f : X \to Y$ in **Set**, $F(f) = id \times f : M \times X \to M \times Y$ which is easily seen to be equivariant. That F is actually a functor and a left adjoint to U is easily checked.$\square$

The above lemma shows that free objects exits in **MSet** and $M \times X$ with the action $s(m, x) = (sm, x)$ is the free M-set on the set X.

2.2 Lemma *The forgetful functor U :* **MSet** $\to$ **Set** *has a right adjoint.*

Proof: Define $H : \mathbf{Set} \to \mathbf{MSet}$ by HX to be the set X^M of all functions from the set M to the set X, with the action of M on X^M given by $(sf)(t) = f(ts)$, for $f \in X^M$ and $s, t \in M$. Also, for a function $h : X \to Y$, define $H(h) : X^M \to Y^M$ by $(Hh)(f) = hf$, for $f \in X^M$. That H is actually a functor and a right adjoint to U is easily checked.$\square$

3 Limits and Colimits in MSet

3.1 Limits Since $\mathbf{MSet} \cong \mathbf{Set}^M$, the category **MSet** is complete and limits are calculated pointwise. In particular, the terminal object of **MSet** is the singleton $\{0\}$, with the obvious M-action. Also, for M-sets A, B, their cartesian product $A \times B$ with the M-action defined by $m(a,b) = (ma, mb)$ is the product of A and B in **MSet**.

3.2 Colimits Since any topos is finitely cocomplete, so is **MSet**. In fact, all colimits in **MSet** exist and are calculated as in **Set** with a natural action of M on them. In particular, $\emptyset$ with the empty action of M on it is the initial object of **MSet**. Also, the coproduct of two M-sets A, B is their disjoint union

$$A \cup B = (A \times \{1\}) \cup (B \times \{2\})$$

with the action of M on $A \cup B$ defined by

$$m(a,1) = (ma,1) \quad , \quad m(b,2) = (mb,2)$$

for $m \in M,\ a \in A,\ b \in B$.

3.3 Monomorphisms Recall that in $\mathbf{Set}^{\mathcal{C}}$, a morphism, that is a natural transformation τ between functors is monic iff each of its components τ_A is monic in **Set**. Therefore, a morphism $\tau = (\tau_M)$ is monic in $\mathbf{Set}^M$ iff its only component τ_M is monic in **Set**. Hence, Since **MSet** is isomorphic to $\mathbf{Set}^M$, a morphism in **MSet** is monic iff it is a monic (one-one) map in **Set**.

3.4 Epimorphisms Consider the adjunction $U \dashv H$, defined in 2.2. Since a left adjoint preserves colimits, the functor U preserves epimorphisms. So, if f is an epimorphism in **MSet** then Uf is an epimorphism in **Set**. But, epimorphisms in **Set** are exactly onto maps. Hence, Uf, and so f, is an onto map. Thus, a morphism in **MSet** is epic iff it is an epic (onto) map in **Set**.

4 Ω in MSet

Here, we explicitly define the subobject classifier in the topos **MSet** and investigate its properties as a lattice.

Recall that, a sieve on the only object M of the category M is a subset S of M which is closed under the left multiplication; i.e. $mx \in S$, for each $m \in M$ and each $x \in S$ (See [9] or [12]). Thus a sieve on M is usually called a **left ideal** of M. Hence, the set $Siv(M)$ of sieves on M, is the set L_M of all the left ideals of M. Note that $\emptyset$ and M are the smallest and the largest ideals of M, respectively.

Thus, the subobject classifier Ω in $\mathbf{Set}^M$ is given by $\Omega(M) = L_M$ and, for $m : M \to M$ (that is $m \in M$),

$$\Omega(m)S = \{x \in M \mid xm \in S\}$$

for $S \in L_M$.

The truth map $t : 1 \to \Omega$ is a natural transformation whose only component $t_M : \{0\} \to L_M$ maps 0 to M, the largest left ideal of M. Also, the false map $f : 1 \to \Omega$ is given by $f_M(0) = \emptyset$, the smallest left ideal of M.

Now, from the isomorphism $\Phi : \mathbf{Set}^M \to \mathbf{MSet}$ one gets that the subobject classifier Ω in **MSet** is L_M with the action of M on it given by "division". That is; for $m \in M$, $S \in L_M$

$$mS = \{x \in M : xm \in S\}.$$

Then we clearly have

i) $m\emptyset = \emptyset$ and $mM = M$, for all $m \in M$;

ii) $eS = S$, for all $S \in \Omega$;

iii) $mS = M$ iff $m \in S$, for each $m \in M$ and $S \in \Omega$;

iv) $S = M$ iff $e \in S$, for each $S \in \Omega$.

4.1 Lemma *A monoid M is a group iff $L_M = \{\emptyset, M\}$.*

Proof: Let M be a group and $S \neq \emptyset$ be a left ideal of M. Take $x \in S$, then $e = x^{-1}x \in S$, and hence $S = M$.

Conversely, let $L_M = \{\emptyset, M\}$ and $e \neq x \in M$. Then $Mx = \{mx : m \in M\}$ is a non-empty left ideal of M. Hence, $Mx = M$. Thus $e \in Mx$. That is $e = mx$, for some $m \in M$. That is x has a left inverse, and hence M is a group.□

4.2 Lemma *Ω has exactly two global elements.*

Proof: For any M-set A, a global element $f : 1 \to A$ is given by an element $k = f(0)$ of A which is fixed under the action of M. For, $mk = mf(0) = f(m0) = f(0) = k$. Now, let $f : 1 \to \Omega$ be a global element of Ω with $f(0) = K \neq \emptyset$. Take $x \in K$. Since K is fixed under the action of M, $xK = K$. That is,

$$K = \{t \in M : tx \in K\}$$

Thus, $e \in xK = K$. Hence $K = M$. This shows that Ω has exactly two global elements t and f, given by $t(0) = M$ and $f(0) = \emptyset$.□

The above lemma says that the topos **MSet** is bivalued.

4.3 Corollary *The topos* **MSet** *is Boolean iff M is a group.*

Proof: By lemma 4.1, M is a group iff $\Omega = \{\emptyset, M\}$ in **MSet**. It is easily shown that the coproduct $1 \coprod 1$ is isomorphic to Ω iff $\Omega = \{\emptyset, M\}$. Thus we get the result.□

5 Ω as a lattice

For any monoid M, the power set $\wp(M)$ of M with the same action of M as given for Ω; that is

$$mB = \{x \in M : xm \in B\}$$

for $m \in M$, $B \subseteq M$, is a left M-set. In fact this M-set is isomorphic to the M-set $H2$ given in lemma 2.2. Further, Ω is a sub-M-set and a sublattice of $\wp(M)$.

In fact, Ω is a Heyting algebra with the operations

$$S \wedge T = S \cap T \ , \ S \vee T = S \cup T \ , \ 0 = \emptyset \ , \ 1 = m \ ,$$

$$S \to T = \{m \in M : mS \subseteq mT\}$$

Thus, Ω is a pseudo-complemented subalgebra of $\wp(M)$. The pseudo-complement of $S \in \Omega$ is given by

$$\begin{array}{ll} S^* = S \to \emptyset & = \{m \in M : mS \subseteq m\emptyset = \emptyset\} \\ & = \{m \in M : (\forall x \in M)(xm \notin S)\} \end{array} .$$

Note that, for any $S \subseteq \Omega$, we have

$$\begin{array}{lll} S^{**} & = \{m \in M & : \ (\forall x \in M)(xm \notin S^*)\} \\ & = \{m \in M & : \ (\forall x \in M)(xmS \not\subseteq \emptyset)\} \\ & = \{m \in M & : \ (\forall x \in M)(\exists y \in M)(yxm \in S)\} \end{array} .$$

The following example shows that the equality $S^{**} \cup S^* = M$ is not true in general, and hence Ω is not necessarily a Stone algebra.

5.1 Example If $M = \{e, a, b\}$ with the operation given by $xy = y$, for $y \neq e$, then $\{a\} \in \Omega$, but

$$\{a\}^* = \{m \in M : m\{a\} = \emptyset\} = \{b\}$$

$$\{a\}^{**} = \{m \in M : m\{b\} = \emptyset\} = \{a\}$$

and so

$$\{a\}^* \cup \{a\}^{**} = \{a, b\} \neq M.$$

We will give conditions on M under which Ω in **MSet** is a Stone algebra. This is a special case of [13], for **MSet**.

5.2 Definition We say that the monoid M satisfies the (left) **Ore condition** if, for every $m, n \in M$, there exist s, $t \in M$ such that $sm = tn$; that is $Mm \cap Mn \neq \emptyset$.

5.3 Proposition *For any monoid M, the following are equivalent in* **MSet.**

i) *M satisfies the left Ore condition.*
ii) *Ω is a Stone algebra.*
iii) *$S^* = \emptyset$, for every non-empty $S \in \Omega$ (one says that Ω is dense).*

Proof: (i) $\Rightarrow$ (ii): Let $S \in \Omega$. If $S = \emptyset$, then clearly $S^{**} \cup S^* = M$. Let $S \neq \emptyset$, and $n \in S$. Let $m \in M$. Then by the hypothesis, for every $x \in M$ there exist $s, t \in M$ such that $s(xm) = tn$. But, since S is a left ideal, $tn \in S$, and hence $sxm \in S$, for all $x \in M$. Thus, by the definition of $S^{**}, m \in S^{**}$. Therefore $S^{**} = M$. Hence Ω is a Stone algebra.

(ii)$\Rightarrow$(iii): Let $\emptyset \neq S \in \Omega$. Since $S^{**} \cup S^* = M$, we get that $e \in S^{**}$ or $e \in S^*$. If $e \in S^*$, then $S = \emptyset$, a contradiction. So $e \in S^{**}$. Thus $S^{**} = M$. Hence, $S^* = S^{***} = M^* = \emptyset$.

(iii)$\Rightarrow$(i): Let $m, n \in M$. Since Mm is a left ideal, by (iii), we get that $(Mm)^* = \emptyset$ and so $n \in (Mm)^{**}$. Hence, by the definition of $(Mm)^{**}$, we have $(xn)Mm \neq \emptyset$, for every $x \in M$. In particular, for $x = e$, $n(Mm) \neq \emptyset$. That is there exists $t \in M$ such that, $tn \in Mm$. So, there exists $s \in M$ such that $tn = sm$.□

5.4 Proposition *The following are equivalent in* **MSet**.
(i) *M satisfies the left Ore condition.*
(ii) *Ω is a Stone algebra.*
(iii) *$S^* = \emptyset$, for all $\emptyset \neq S \in \Omega$.*
(iv) *$S \cap T = \emptyset$ implies $S = \emptyset$ or $T = \emptyset$, for $S, T \in \Omega$.*
(v) *$(S \cap T)^* = S^* \cup T^*$, for $S, T \in \Omega$.*
(vi) *$(S \cup T)^{**} = S^{**} \cup T^{**}$, for $S, T \in \Omega$.*
(vii) *$Rg(\Omega) = \{S^* : S \in \Omega\}$ is a sublattice of Ω.*

6 Exponentiation in MSet

In this finall section, we discuss the exponentiation in the topos **MSet**. Recall that for F, G in $\mathbf{Set}^{\mathcal{C}}$, G^F is defined by $G^F(U) = Hom(h_U \times F, G)$, for an object U of $\mathcal{C}$, and for a morphism $\alpha : U \to V$, $G^F(\alpha) = Hom(h_\alpha \times id, G)$ which maps each $\eta : h_U \times F \to G$ to $\eta \circ (h_\alpha \times id_F)$, where h_U and $h_\alpha : h_V \to h_U$ as the usual ones. So, in prticular, for $\mathcal{C} = M$, $G : M \to \mathbf{Set}$ maps the only object of M to $Hom(h_M \times F, G)$, where $h_M(M) = M$ and, for each $m \in M$, $G(m) = Hom(h_m \times id, G)$ maps $\eta : h_M \times F \to G$ to $\alpha = \eta \circ (h_m \times id)$ which is a natural transformation with only one component $\alpha_M : M \times FM \to GM$, given by $\alpha_M(s, x) = \eta_M(sm, x)$. Now, by the isomorphism $\Phi : \mathbf{Set}^M \cong \mathbf{MSet}$, for M-sets A, B, we have

$$B^A = Hom_{\mathbf{MSet}}(M \times A, B)$$

with the action given by

$$(mf)(s,a) = f(sm,a)$$

for $m \in M,\ f \in B^A$.

Now we show that B^A is actually the exponential of A and B in **MSet**.

6.1 Proposition *For any A, B in* **MSet**, *B^A as defined above is the exponentiation of B to A in* **MSet**.

Proof: To prove that the functor $(-)^A : \mathbf{MSet} \to \mathbf{MSet}$ is a right adjoint to the functor $- \times A : \mathbf{MSet} \to \mathbf{MSet}$, it is enough to see that

$$Hom_{\mathbf{MSet}}(C \times A, B) \cong Hom_{\mathbf{MSet}}(C, B^A)$$

for every M-sets A, B, C. Define

$$\alpha : Hom(C \times A, B) \to Hom(C, B^A)$$

by $[\alpha(g)(x)](s,a) = g(sx,a)$, for $g \in Hom(C \times A, B), x \in C, s \in M, a \in A$; and

$$\beta : Hom(C, B^A) \to Hom(C \times A, B)$$

by $\beta(f)(x,a) = f(x)(e,a)$, for $f \in Hom(C, B^A), x \in C, a \in A$. Then α, β are inverse of each other. The naturality in C, A, B is obvious.□

6.2 Corollary *For any M-set B, we have*

$$Hom_{\mathbf{MSet}}(M, B) \cong B$$

6.3 Corollary *For any A in* **MSet**, *Ω^A is isomorphic to Sub $(M \times A)$, the set of all subobjects of $M \times A$ in* **MSet.**

Proof: By the above proposition and the property of Ω, we have

$$\Omega^A = Hom_{\mathbf{MSet}}(M \times A, \Omega) \cong Sub(M \times A)$$

where the above isomorphism is a bijection which can be made into an isomorphism in **MSet**, by defining the action of M on $Sub(M \times A)$ as below:

$$sX = \{(m,a) : (ms,a) \in X\}$$

for $s \in M,\ X \in Sub(M \times A)$.□

6.4 Remark Let X be a subobject of $M \times A$. Then X is a subset of $M \times A$ which is closed under the M-action. X, being a subset of $M \times A$, can be written as

$$X = \bigcup\nolimits_{m \in M} \{m\} \times X_m$$

where $X_m = \{a \in A : (m, a) \in X\}$. Since X is closed under the M-action, we have

$$(m, a) \in X \Rightarrow (sm, sa) = s(m, a) \in X$$

for every $s \in M$. That is, for every $s \in M$,

$$a \in X_m \Rightarrow sa \in X_{sm}$$

Thus, X can be identified by a family $(X_m)_{m \in M}$ where, for each $m \in M$, X_m is a subset of A with

$$(\forall s \in M)(a \in X_m \Rightarrow sa \in X_{sm})$$

which is equivalent to $X = (X_m)_{m \in M}$ being in Ω iff

$$(\forall s \in M)(sX_m \subseteq X_{sm})$$

where $sX_m = \{sx : x \in X_m\}$. With this identification, the action of M on Ω^A is given by

$$sX = (X_{ms})_{m \in M}$$

6.5 Remark If M is a group, and $X = (X_m)_{m \in M} \in \Omega^A$ then $sX_m = X_{sm}$, for $s, m \in M$. For, if $a \in X_{sm}$ then $(sm, a) \in X$. So, $(m, s^{-1}a) = s^{-1}(sm, a) \in X$. Thus $s^{-1}a \in X_m$ and hence $a = s(s^{-1}a) \in sX_m$. Therefore, $X_{sm} \subseteq sX_m$. The converse follows by the above remark. This, in particular, shows that $X_s = sX_e$, for every $s \in M$. Thus, any X in Ω^A is completely determined by $X_e = \{a \in A : (e, a) \in X\}$.

The following lemma can easily be proved.

6.6 Lemma *For any M-set A, Ω^A is a bounded lattice, with the operations defined componentwise; i.e.*

$$(X_m)_{m \in M} \vee (Y_m)_{m \in M} = (X_m \cup Y_m)_{m \in M}$$

$$1 = (A_s)_{s \in M}; \text{ where } A_s = A, \forall s \in M$$

$$0 = (\emptyset_s)_{s \in M}; \text{ where } \emptyset_s = \emptyset, \forall s \in M$$

$$(X_m)_{m \in M} \wedge (Y_m)_{m \in M} = (X_m \cap Y_m)_{m \in M}. \square$$

6.7 Lemma *If M is a group then, for any M-sets A and B, B^A is isomorphic to $Hom_{\mathbf{Set}}(A, B)$ with the action $(mg)(a) = mg(m^{-1}a)$, for any function $g : A \to B$ and $m \in M, a \in A$.*

Proof: We know that $B^A = Hom_{\mathbf{MSet}}(M \times A, B)$. Define

$$\alpha : Hom_{\mathbf{MSet}}(M \times A, B) \to Hom_{\mathbf{Set}}(A, B)$$

by $\alpha(f)(a) = f(e, a)$, and

$$\beta : Hom_{\mathbf{Set}}(A, B) \to Hom_{\mathbf{MSet}}(M \times A, B)$$

by $\beta(g)(m, a) = mg(m^{-1}a)$. The fact that α and β are equivariant, and $\alpha \circ \beta = id$, $\beta \circ \alpha = id$ is easily checked.□

6.8 Corollary *If M is a group and A an M-set, then Ω^A is isomorphic to $\wp(A)$, where the action on $\wp(A)$ is given by $mY = \{ma : a \in Y\}$, for $m \in M$ and $Y \subseteq A$.*

Proof: By the above lemma, $\Omega^A \cong Hom_{\mathbf{MSet}}(A, \Omega)$. But, since M is a group, $\Omega \cong 2$. Hence

$$\Omega^A \cong Hom_{\mathbf{Set}}(A, 2) \cong \wp(A).$$

In fact, this isomorphism maps $X = (X_m)_{m \in M}$ in Ω^A to X_e, and is clearly equivariant.□

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