Research project on Weil-étale cohomology

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In this note I will briefly describe what is Weil-étale cohomology, my contributions to its study, and plans for further research.

1 Zeta functions of schemes

Given a scheme X of finite type over $\operatorname{Spec}\mathbb{Z}$, one may attach to it the corresponding **zeta function**

$$\zeta_X(s) = \prod_{x \in |X|} \frac{1}{1 - N(x)^{-s}}.$$

Here $|X|$ denotes the set of closed points of X, and for $x \in |X|$ the norm $N(x)$ is the order of the corresponding residue field $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$. The above product converges for $\operatorname{Re} s > \dim X$, and it is conjectured that the zeta function admits a meromorphic continuation to the whole complex plane.

In particular, if $X = \operatorname{Spec}\mathcal{O}_F$ is the spectrum of the ring of integers of a number field $F/\mathbb{Q}$, then $\zeta_X(s) = \zeta_F(s)$ is the Dedekind zeta function, which has been extensively studied in algebraic number theory. If X is a smooth projective variety over a finite field $\mathbb{F}_q$, then $\zeta_X(s) = Z_X(q^{-s})$, where

$$Z_X(t) = \exp\Bigl(\sum_{n\ge 1} \frac{\# X(\mathbb{F}_{q^n})}{n}\, t^n\Bigr)$$

is the Hasse–Weil zeta function, whose basic properties are given by Weil conjectures in algebraic geometry (see e.g. [Kat1994]).

For basic facts and conjectures about zeta functions of schemes I refer to Serre's survey [Ser1965].

Of particular interest are the so-called **special values** of $\zeta_X(s)$ at integers $s = n \in \mathbb{Z}$. Namely, if d_n is the vanishing order of $\zeta_X(s)$ at $s = n$, then the corresponding special value is defined as the leading Taylor coefficient at $s = n$:

$$\zeta_X^*(n) = \lim_{s\to n} (s-n)^{d_n}\, \zeta_X(s)$$

(assuming the meromorphic continuation around $s = n$).

2 Weil-étale cohomology

It is expected that both the vanishing order d_n and the special value $\zeta_X^*(n)$ can be expressed by *certain invariants* of X. There are many conjectures of varying generality that make this precise. I am interested in a relatively recent **Weil-étale cohomology** program, initiated by Stephen Lichtenbaum [Lic2005, Lic2009a, Lic2009b]. Other results for the case of varieties over finite fields were obtained by Geisser [Gei2004, Gei2006].

Let me briefly explain what one expects from Weil-étale cohomology. Let X be a separated scheme of finite type over $\operatorname{Spec}\mathbb{Z}$. Then for a fixed integer n, the Weil-étale cohomology consists of abelian groups $H^i_{W,c}(X,\mathbb{Z}(n))$ with the following conjectural properties.

W1) $H^i_{W,c}(X,\mathbb{Z}(n))$ are finitely generated abelian groups, trivial for $|i| \gg 0$.

Consequently, one can assign to these groups the corresponding **determinant** $\det_{\mathbb{Z}} H^\bullet_{W,c}(X,\mathbb{Z}(n))$ in the sense of Knudsen and Mumford [KM1976], which is a free $\mathbb{Z}$-module of rank 1.

W2) After tensoring these cohomology groups with $\mathbb{R}$, one obtains a long exact sequence of finite-dimensional real vector spaces

$$\cdots \to H^{i-1}_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R} \xrightarrow{\smile\theta} H^{i}_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R} \xrightarrow{\smile\theta} H^{i+1}_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R} \to \cdots$$

By well-known properties of determinants of complexes, this induces a *canonical* isomorphism

$$\lambda\colon \mathbb{R} \xrightarrow{\cong} \Big(\det_{\mathbb{Z}} H^{\bullet}_{W,c}(X,\mathbb{Z}(n))\Big)\otimes\mathbb{R}.$$

W3) The vanishing order of $\zeta_X(s)$ at $s = n \in \mathbb{Z}$ is conjecturally given by

$$d_n = \sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot \operatorname{rk}_{\mathbb{Z}} H^i_{W,c}(X,\mathbb{Z}(n)).$$

W4) The corresponding special value is determined up to sign by

$$\lambda(\zeta^*_X(n)^{-1})\cdot\mathbb{Z} = \det_{\mathbb{Z}} H^{\bullet}_{W,c}(X,\mathbb{Z}(n)).$$

Baptiste Morin gave in [Mor2014] a construction of Weil-étale cohomology for X a separated scheme of finite type, proper and regular, and $n = 0$. Later this construction was generalized together with Matthias Flach in [FM2018] to any $n \in \mathbb{Z}$, under the same assumptions on X (for $n > 0$ the formula W4) is corrected by a rational factor $C(X,n)$, defined in [ibid. §5.4]).

3 My work on Weil-étale cohomology

In my doctoral thesis [Besh-0], which was co-supervised by Baptiste Morin and Bas Edixhoven, I generalized the work of Flach and Morin to any X that is separated and of finite type over $\operatorname{Spec}\mathbb{Z}$ (thus removing the assumption that X is proper or regular) for the case of $n < 0$ (which, as it turns out, simplifies certain aspects of the theory).

The main building block of $H^i_{W,c}(X,\mathbb{Z}(n))$ is the (étale) **motivic cohomology** $H^i_{ét}(X,\mathbb{Z}^c(n))$, defined in terms of "dualizing cycle complexes" $\mathbb{Z}^c(n)$, as considered by Geisser in [Gei2010] in the context of **arithmetic duality theorems**.

The exact constructions are quite technical, so I refer the reader to my preprints [Besh-1, Besh-2] for more details. It is important to note that

- the above property W1) is established assuming the finite generation of étale motivic cohomology $H^i_{ét}(X,\mathbb{Z}^c(n))$;
- the sequence of real vector spaces in W2) is defined via a **regulator** map; the exactness in W2) follows from Beilinson conjectures about regulators.

Like most of the formulas for special values, this is all conjectural, especially at the level of generality we are interested in. A compelling evidence in favor of the above special value conjecture is that, whenever the comparison makes sense, it is equivalent to the **Tamagawa number conjecture** (**TNC**) of Bloch–Kato–Fontaine–Perrin-Riou [FPR1994].

A new interesting point is the following. If $Z \subset X$ is a closed subscheme and $U = X \setminus Z$ is its open complement, then $\zeta_X(s) = \zeta_Z(s)\zeta_U(s)$. Accordingly, one should expect the special value conjecture to be compatible with such "closed-open decompositions" of schemes: the special value conjecture for X should be equivalent to the corresponding conjecture for Z and U. I prove in [Besh-2] that this is indeed the case in my situation: morally, this follows from a long exact sequence

$$\cdots \to H^i_{W,c}(U,\mathbb{Z}(n)) \to H^i_{W,c}(X,\mathbb{Z}(n)) \to H^i_{W,c}(Z,\mathbb{Z}(n)) \to H^{i+1}_{W,c}(U,\mathbb{Z}(n)) \to \cdots$$

Similarly, it is not hard to see that one has $\zeta_{\mathbb{A}^r_X}(s) = \zeta_X(s-r)$, and hence a special value conjecture for the affine bundle $\mathbb{A}^r_X$ at $s = n$ should be equivalent to the corresponding conjecture for X at $s = n - r$. I prove in [Besh-2] that this is also the case.

As a result, we can start from certain very special cases of X for which the special value conjecture is known (e.g. using equivalence with TNC and known cases of TNC), and then construct new schemes (not necessarily proper or regular) for which the conjecture also holds, unconditionally. I refer the reader to [Besh-2] for precise unconditional results.

In a recent preprint [Besh-3], I was able to compute the Weil-étale cohomology of an arbitrary one-dimensional separated scheme of finite type $X \to \operatorname{Spec} \mathbb{Z}$ and $n < 0$, and as a consequence, obtain a new formula for the special values $\zeta_X^*(n)$ for $n < 0$ in terms the étale motivic cohomology of X and a regulator. This was motivated in part by the work of Jordan and Poonen, who worked out a formula for $\zeta_X^*(1)$ for X one-dimensional, reduced and affine in [JP2020]. My formula from [Besh-3] holds unconditionally if for all generic points $\eta \in X$ with $\operatorname{char} \kappa(\eta) = 0$ the corresponding extension $\kappa(\eta)/\mathbb{Q}$ is abelian.

4 My research plans

Here I will list several problems that I have in mind for my future work.

1. It is an interesting project to extend the results of [Besh-3] to all integers $n \in \mathbb{Z}$.

   In the work of Flach and Morin [FM2018] the corresponding special value conjecture for $n > 0$ involves a "correction factor" $C(X,n) \in \mathbb{Q}$. Subsequent work [FM2020a, FM2020b] sheds some light on its nature. The key to generalizing Flach and Morin's work for non-regular or non-proper X seems to lie in understanding the required correction factor. The recent work of Morin [Mor2021b] may be particularly helpful here.

2. Lichtenbaum in his recent preprint [Lic2021] gives a different special value conjecture, which up to a power of 2 shold be equivalent to that of Flach and Morin [FM2018]. It would be useful to explicitly compare the two conjectures.

3. It is likely that my results can be generalized to Weil-étale cohomology for $\mathbb{Z}$-constructible sheaves $\mathscr{F}$, in the spirit of [GS2020] and [Mor2021a]. I plan to consider these generalizations in the near future.

4. The construction of Weil-étale cohomolgy by Flach and Morin [FM2018] uses the so-called **Artin–Verdier topology**, a modification of the étale topology $X_{\text{ét}}$, which is supposed to take care of the real places $X(\mathbb{R})$. A suitable construction is given in [ibid., §6], but *only for proper and regular* X. The methods I use in [Besh-1] circumvent this restriction, but at the cost of introducing an auxiliary cohomology theory $H^i_{fg}(X, \mathbb{Z}(n))$ which is not bounded, but can instead have finite 2-torsion in arbitrarily high degrees $i \gg 0$.

   It would be interesting to work out the right construction of Artin–Verdier topology for any $X \to \operatorname{Spec} \mathbb{Z}$ separated and of finite type, and rework [Besh-1] accordingly.

5. The actual construction of Weil-étale cohomology is not formulated in terms of separate groups $H^i_{W,c}(X, \mathbb{Z}(n))$, but in terms of complexes $R\Gamma_{W,c}(X, \mathbb{Z}(n))$. At present, these are defined (both in [FM2018] and in [Besh-1]) up to a *non-unique* isomorphism in the derived category $\mathbf{D}(\mathbb{Z})$, as a mapping fiber of certain canonical morphism in $\mathbf{D}(\mathbb{Z})$. This is not much of a problem for the special value conjecture, since the determinants $\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)) = \det_{\mathbb{Z}} H^\bullet_{W,c}(X, \mathbb{Z}(n))$ are uniquely defined, but it would be useful to find a more canonical definition for $R\Gamma_{W,c}(X, \mathbb{Z}(n))$.

   This could probably be remedied using dg-categories [Toë11] or stable ∞-categories [Lur2009].

6. In the above exposition, I have swept some details about regulators under the rug. I use the construction of Kerr, Lewis, and Müller-Stach from [KLMS2006]. In general, the regulator has to do with complex points $X(\mathbb{C})$, and the regulator in [KLMS2006] is defined for smooth and quasi-projective $X_{\mathbb{C}}$. This is rather unfortunate, because my construction of Weil-étale cohomology $H^i_{W,c}(X, \mathbb{Z}(n))$ works for any X that is separated and of finite type over $\operatorname{Spec} \mathbb{Z}$ (assuming finite generation of the corresponding motivic cohomology). It would be interesting to find suitable generalizations of the regulator for singular $X_{\mathbb{C}}$, and connect them to my machinery.

There are variants of motivic cohomology for singular complex varieties, defined in terms of hyperresolutions, e.g. Hanamura's "Chow cohomology groups" [Han2000], which allow us to formally extend the formula of Kerr–Lewis–Müller-Stach to the singular case. However, it seems that singular regulators have not been thoroughly considered for special value conjectures.

7. The Tamagawa number conjecture has a generalization, known as the **equivariant Tamagawa number conjecture** (**ETNC**); see e.g. Flach's survey [Fla2004]. Similarly, Weil-étale cohomology should also have an "equivariant refinement", and it would be interesting to write this down for my construction and prove the compatibility with ETNC.

These are a few specific questions that arise naturally from my previous work. In general, Weil-étale cohomology is an active research topic and there are still many open problems in this context.

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