

WEIL-ÉTALE COHOMOLOGY AND DUALITY FOR ARITHMETIC SCHEMES IN NEGATIVE WEIGHTS

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Abstract

Flach and Morin constructed in [9] Weil-étale cohomology $H_{W,c}^i(X, \mathbb{Z}(n))$ for a proper, regular arithmetic scheme X (i.e. separated and of finite type over $\text{Spec } \mathbb{Z}$) and $n \in \mathbb{Z}$. In the case when $n < 0$, we generalize their construction to an arbitrary arithmetic scheme X , thus removing the proper and regular assumption. The construction assumes finite generation of suitable étale motivic cohomology groups.

1 Introduction

Stephen Lichtenbaum, in a series of papers [26, 27, 28], has envisioned a new cohomology theory for schemes, known as **Weil-étale cohomology**. The case of varieties over finite fields X/\mathbb{F}_q was further studied by Geisser [11, 13, 14]. Morin defined in [34] Weil-étale cohomology with compact support $H_{W,c}^i(X, \mathbb{Z})$ for $X \rightarrow \text{Spec } \mathbb{Z}$ separated, of finite type, proper, and regular. This construction was further generalized by Flach and Morin in [9] to the groups $H_{W,c}^i(X, \mathbb{Z}(n))$ with arbitrary weights $n \in \mathbb{Z}$, under the same assumptions on X .

The aim of this paper is to remove the assumption that X is proper and regular and, following the ideas of [9], to construct the groups $H_{W,c}^i(X, \mathbb{Z}(n))$ for any X separated and of finite type over $\text{Spec } \mathbb{Z}$ for the case of strictly negative weights $n < 0$.

As Flach and Morin already suggest in [9, Remark 3.11], we rework all their constructions in terms of cycle complexes $\mathbb{Z}^c(n)$, which were considered by Geisser in [15] in the context of arithmetic duality theorems.

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In a forthcoming paper we apply the results of this text to relate the cohomology groups $H_{W,c}^i(X, \mathbb{Z}(n))$ to the special value of the zeta function $\zeta(X, s)$ at $s = n < 0$.

Notation and conventions

Arithmetic schemes. In this work, an **arithmetic scheme** is a scheme X that is separated and of finite type over $\text{Spec } \mathbb{Z}$.

Abelian groups. Let A be an abelian group. For $m \geq 1$ we denote by ${}_m A$ its m -torsion subgroup, and by A_m the quotient A/mA :

$$0 \rightarrow {}_m A \rightarrow A \xrightarrow{\times m} A \rightarrow A_m \rightarrow 0$$

We denote by A_{div} (resp. A_{tor}) the maximal divisible subgroup (resp. maximal torsion subgroup), and by A_{cotor} the quotient A/A_{tor} .

We say that A is of **cofinite type** if it is \mathbb{Q}/\mathbb{Z} -dual to a finitely generated abelian group: $A = \text{Hom}(B, \mathbb{Q}/\mathbb{Z})$ for a finitely generated B .

Complexes. All our constructions take place in the derived category of abelian groups $\mathbf{D}(\mathbb{Z})$. For our purposes, we introduce the following terminology. Recall first that a complex of abelian groups A^\bullet is **perfect** if it is bounded (i.e. $H^i(A^\bullet) = 0$ for $|i| \gg 0$), and $H^i(A^\bullet)$ are finitely generated abelian groups.

DEFINITION 1.1. A complex of abelian groups A^\bullet is **almost perfect** if the cohomology groups $H^i(A^\bullet)$ are finitely generated, and bounded, except for possible finite 2-torsion in arbitrarily high degree. That is, $H^i(A^\bullet) = 0$ for $i \ll 0$ and $H^i(A^\bullet)$ is finite 2-torsion for $i \gg 0$.

A complex of abelian groups A^\bullet is of **cofinite type** if the cohomology groups $H^i(A^\bullet)$ are of cofinite type and bounded.

A complex of abelian groups A^\bullet is **almost of cofinite type** if the cohomology groups $H^i(A^\bullet)$ are of cofinite type and bounded, except for possible finite 2-torsion in arbitrarily high degree.

This terminology is ad hoc and was invented for this text, since such complexes will appear frequently. Some basic observations about almost perfect and almost cofinite type complexes are collected in Appendix A. We note that this finite 2-torsion in arbitrarily high degrees could be removed by working with the Artin–Verdier topology $\overline{X}_{\acute{e}t}$ instead of the usual étale topology $X_{\acute{e}t}$. The general construction and basic properties of $\overline{X}_{\acute{e}t}$ are treated in [9, Appendix A], but only for a *proper and regular* arithmetic scheme X . Our methods circumvent this restriction at the cost of some technical hurdles with 2-torsion.

Étale cohomology. For an arithmetic scheme X and a complex of étale sheaves \mathcal{F}^\bullet , we denote by

$$R\Gamma(X_{\text{ét}}, \mathcal{F}^\bullet) \text{ (resp. } R\Gamma_c(X_{\text{ét}}, \mathcal{F}^\bullet), R\widehat{\Gamma}_c(X_{\text{ét}}, \mathcal{F}^\bullet))$$

the complex that computes the corresponding cohomology, resp. cohomology with compact support, and modified cohomology with compact support. For the convenience of the reader, we review the definitions in Appendix B. The purpose of $R\widehat{\Gamma}_c(X_{\text{ét}}, \mathcal{F}^\bullet)$ is to take care of real places $X(\mathbb{R})$. There exists a canonical projection $R\widehat{\Gamma}_c(X_{\text{ét}}, \mathcal{F}^\bullet) \rightarrow R\Gamma_c(X_{\text{ét}}, \mathcal{F}^\bullet)$, which is an isomorphism if $X(\mathbb{R}) = \emptyset$.

G -equivariant sheaves and their cohomology. Let \mathcal{X} be a topological space with an action of a discrete group G . A **G -equivariant sheaf** \mathcal{F} on \mathcal{X} can be defined as an espace étalé $\pi: E \rightarrow \mathcal{X}$ with a G -action on E such that the projection π is G -equivariant (see e.g. [30, §II.6 + pp. 594]). We denote by $\mathbf{Sh}(G, \mathcal{X})$ the corresponding category.

The equivariant global sections are defined by

$$\Gamma(G, \mathcal{X}, \mathcal{F}) = \mathcal{F}(\mathcal{X})^G,$$

with G acting on $\mathcal{F}(\mathcal{X}) = \{s: \mathcal{X} \rightarrow E \mid \pi \circ s = \text{id}_{\mathcal{X}}\}$ via $(g \cdot s)(x) = g \cdot s(g^{-1} \cdot x)$. The corresponding **G -equivariant cohomology** is given by the right derived functors of $\Gamma(G, \mathcal{X}, -)$.

More details on G -equivariant sheaves can be found in [33, Chapitre 2]. For our modest purposes, it suffices to know that any G -module A gives rise to the corresponding abelian G -equivariant constant sheaf. The latter corresponds to the espace étalé $\mathcal{X} \times A \rightarrow \mathcal{X}$, where A is endowed with the discrete topology.

$G_{\mathbb{R}}$ -equivariant cohomology of $X(\mathbb{C})$. Given an arithmetic scheme X , we denote by $X(\mathbb{C})$ the set of complex points of X endowed with the analytic topology. It carries the natural action of the Galois group $G_{\mathbb{R}} := \text{Gal}(\mathbb{C}/\mathbb{R})$.

We consider the $G_{\mathbb{R}}$ -modules

$$\mathbb{Z}(n) := (2\pi i)^n \mathbb{Z}, \quad \mathbb{Q}(n) := (2\pi i)^n \mathbb{Q}, \quad \mathbb{Q}/\mathbb{Z}(n) := \mathbb{Q}(n)/\mathbb{Z}(n)$$

as constant $G_{\mathbb{R}}$ -equivariant sheaves on $X(\mathbb{C})$.

Then $R\Gamma_c(X(\mathbb{C}), A(n))$ for $A = \mathbb{Z}, \mathbb{Q}, \mathbb{Q}/\mathbb{Z}$ (the complex that computes singular cohomology with compact support of $X(\mathbb{C})$ with coefficients in $A(n)$) is a complex of $G_{\mathbb{R}}$ -modules, and we can further take the group cohomology (resp. Tate cohomology):

$$\begin{aligned} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), A(n)) &:= R\Gamma(G_{\mathbb{R}}, R\Gamma_c(X(\mathbb{C}), A(n))), \\ R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), A(n)) &:= R\widehat{\Gamma}(G_{\mathbb{R}}, R\Gamma_c(X(\mathbb{C}), A(n))). \end{aligned}$$

By definition, this is the **$G_{\mathbb{R}}$ -equivariant cohomology** (resp. **$G_{\mathbb{R}}$ -equivariant Tate cohomology**) with compact support of $X(\mathbb{C})$ with coefficients in $A(n)$.

Motivic cohomology $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$. Our construction is based on motivic cohomology defined in terms of complexes of sheaves $\mathbb{Z}^c(n)$ on $X_{\acute{e}t}$. We follow the notation of [15].

Briefly, for $i \geq 0$ we consider the algebraic simplex

$$\Delta^i = \text{Spec } \mathbb{Z}[t_0, \dots, t_i] / (\sum_i t_i - 1).$$

We fix a negative weight $n \leq 0$. Let $z_n(X, i)$ be the free abelian group generated by the closed integral subschemes $Z \subset X \times \Delta^i$ of dimension $n+i$ that intersect the faces properly. Then $z_n(X, \bullet)$ is a (homological) complex of abelian groups whose differentials are given by the alternating sum of intersections with the faces. We consider the (cohomological) complex of étale sheaves

$$\mathbb{Z}^c(n) := z_n(_, -\bullet)[2n].$$

The boundedness from below of $\mathbb{Z}^c(n)$ is not known in general; it is a variant of the Beilinson–Soulé vanishing conjecture. To work unconditionally with the derived functors, we use K -injective resolutions [38, 36] (resp. K -flat resolutions for the derived tensor products).

To avoid any confusion, we use cohomological numbering for all complexes in this paper, so we set

$$H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) := H^i(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n))).$$

([15] uses homological numbering.)

If X is *proper, regular and of pure dimension d* , then for $n \leq 0$ there exists an isomorphism

$$(1) \quad H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) \cong H^{2d+i}(X_{\acute{e}t}, \mathbb{Z}(d-n)),$$

where the right-hand side is the “usual” motivic cohomology defined for positive weights; see the original Bloch’s paper [4] for the case of varieties, and also [10, 12] for the definitions and properties over $\text{Spec } \mathbb{Z}$.

Assumptions

Weights. In this paper, $n < 0$ always denotes a strictly negative integer, which will be the weight in the cohomology groups $H_{W,c}^i(X, \mathbb{Z}(n))$.

Finite generation conjecture. Our construction of the Weil-étale cohomology groups $H_{W,c}^i(X, \mathbb{Z}(n))$ uses the following assumption.

CONJECTURE 1.2. $\mathbf{L}^c(X_{\acute{e}t}, n)$: *for an arithmetic scheme X and $n < 0$, the cohomology groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are finitely generated for all $i \in \mathbb{Z}$.*

See Proposition 8.3 for the precise relation of $\mathbf{L}^c(X_{\acute{e}t}, n)$ to other conjectures that appear in the literature. We refer to §8 for the cases where the conjecture is known.

Main results

Before outlining the construction of Weil-étale cohomology, we state the main results of this paper that make it possible. One of our main objects is the following complex of abelian sheaves $\mathbb{Z}(n)$ on $X_{\acute{e}t}$.

DEFINITION 1.3 ([9, §3.1], [11, §7]). Let X be an arithmetic scheme and $n < 0$. For a prime p , consider the localization $X[1/p]$, and let μ_{p^r} be the sheaf of p^r -th roots of unity on $X[1/p]$. We define the twist of μ_{p^r} by n as

$$\mu_{p^r}^{\otimes n} = \underline{\mathrm{Hom}}_{X[1/p]}(\mu_{p^r}^{\otimes(-n)}, \mathbb{Z}/p^r\mathbb{Z}).$$

Now $\mathbb{Z}(n)$ is the complex of sheaves on $X_{\acute{e}t}$ given by

$$\mathbb{Z}(n) = \mathbb{Q}/\mathbb{Z}(n)[-1], \quad \text{where } \mathbb{Q}/\mathbb{Z}(n) = \bigoplus_p \varinjlim_r j_{p!} \mu_{p^r}^{\otimes n},$$

and j_p is the canonical open immersion $X[1/p] \rightarrow X$.

The above sheaves $\mathbb{Z}(n)$ should not be confused with cycle complexes; the latter are $\mathbb{Z}^c(n)$ in the context of this paper. In §2 we prove the following arithmetic duality theorem relating the two.

Theorem I. *Assuming Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, there is a quasi-isomorphism*

$$R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n)) \xrightarrow{\cong} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]).$$

The second result we need is related to the following morphism of complexes.

DEFINITION 1.4. We define $v_\infty^* : R\Gamma_c(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}(n)) \rightarrow R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))$ as the morphism in the derived category $\mathbf{D}(\mathbb{Z})$ induced by the comparison of étale and analytic topology

$$\Gamma_c(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}(n)) \rightarrow \Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \alpha^* \mathbb{Q}/\mathbb{Z}(n)) \cong \Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))$$

(see Proposition B.5 and 6.1). Then we let $u_\infty^* : R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \rightarrow R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))$ be the composition

$$\begin{aligned} R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) &:= R\Gamma_c(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}(n))[-1] \xrightarrow{v_\infty^*[-1]} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))[-1] \\ &\rightarrow R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \end{aligned}$$

where the last arrow is induced by $\mathbb{Q}/\mathbb{Z}(n)[-1] \rightarrow \mathbb{Z}(n)$, which comes from the distinguished triangle of constant $G_{\mathbb{R}}$ -equivariant sheaves $\mathbb{Z}(n) \rightarrow \mathbb{Q}(n) \rightarrow \mathbb{Q}/\mathbb{Z}(n) \rightarrow \mathbb{Z}(n)[1]$.

Then §6 is devoted to the following result.

Theorem II. *The morphism u_∞^* is torsion, i.e. there exists a nonzero integer m such that $mu_\infty^* = 0$*

Sketch of the construction of Weil-étale cohomology

Here we describe the structure of this paper, as well as our construction of the Weil-étale complexes $R\Gamma_{W,c}(X, \mathbb{Z}(n))$.

First, §2 is devoted to the proof of Theorem I. Some of its consequences are deduced in §4. Namely, if we assume Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, then $R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is an almost perfect complex, while $R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n))$ is almost of cofinite type in the sense of Definition 1.1. For this, we first make a small digression in §3 to analyze what kind of complexes we obtain for the $G_{\mathbb{R}}$ -equivariant cohomology of $X(\mathbb{C})$.

Theorem I is used in §5 to define a morphism $\alpha_{X,n}$ in the derived category (see Definition 5.1), and declare $R\Gamma_{fg}(X, \mathbb{Z}(n))$ to be its cone:

$$\begin{aligned} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) &\xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \rightarrow R\Gamma_{fg}(X, \mathbb{Z}(n)) \\ &\rightarrow R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) \end{aligned}$$

The notation “ fg ” comes from the fact that $R\Gamma_{fg}(X, \mathbb{Z}(n))$ is an almost perfect complex in the sense of Definition 1.1. Thanks to specific properties of the complexes involved, it turns out that $R\Gamma_{fg}(X, \mathbb{Z}(n))$ is defined up to a *unique* isomorphism in the derived category (which is not normally expected from a cone).

Then in §6 we establish Theorem II, and it is used in §7 to define Weil-étale complexes $R\Gamma_{W,c}(X, \mathbb{Z}(n))$. To do this, we deduce from Theorem II that $u_\infty^* \circ \alpha_{X,n} = 0$, which implies that there exists a morphism in the derived category

$$i_\infty^* : R\Gamma_{fg}(X, \mathbb{Z}(n)) \rightarrow R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))$$

—see (2) below. We choose a mapping fiber of i_∞^* and call it $R\Gamma_{W,c}(X, \mathbb{Z}(n))$, which turns out to be a perfect complex. Finally, in §8 we consider the cases of X for which Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ is known, and hence our results hold unconditionally, and in §9 we verify that if X is proper and regular, our complex $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ is isomorphic to that constructed in [9] by Flach and Morin.

There are two appendices to this paper: Appendix A collects some lemmas from homological algebra, and Appendix B gives an overview of the definitions of étale cohomology with compact support $R\Gamma_c(X_{\acute{e}t}, -)$ and its modified version $R\widehat{\Gamma}_c(X_{\acute{e}t}, -)$.

The definition of $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ fits in the following commutative diagram with distinguished triangles in the derived category $\mathbf{D}(\mathbb{Z})$:

$$(2) \quad \begin{array}{ccccc} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & \longrightarrow & 0 & & \\ \mathrm{Dfn. 5.1} \downarrow \alpha_{X,n} & & \downarrow & & \\ R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) & \xrightarrow[\mathrm{Dfn. 1.4}]{u_\infty^*} & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) & & \\ \downarrow & & \downarrow \mathrm{id} & & \\ R\Gamma_{W,c}(X, \mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X, \mathbb{Z}(n)) & \xrightarrow{i_\infty^*} & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \longrightarrow R\Gamma_{W,c}(X, \mathbb{Z}(n))[1] \\ \downarrow & & \downarrow & & \downarrow \\ R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) & \longrightarrow & 0 & & \end{array}$$

Our construction follows [9], and the resulting complex is the same if X is proper and regular, which is the assumption considered by Flach and Morin. Here is a brief comparison between the notations.

this paper	Flach–Morin
$X \rightarrow \mathrm{Spec} \mathbb{Z}$	$X \rightarrow \mathrm{Spec} \mathbb{Z}$
separated, of finite type	separated, of finite type proper, regular, equidimensional
$n < 0$	$n \in \mathbb{Z}$
cycle complexes $\mathbb{Z}^c(n)$	cycle complexes $\mathbb{Z}(d-n)[2d]$, $d = \dim X$
$R\Gamma_{fg}(X, \mathbb{Z}(n))$	$R\Gamma_W(\bar{X}, \mathbb{Z}(n))$, up to finite 2-torsion
$R\Gamma_{W,c}(X, \mathbb{Z}(n))$	$R\Gamma_{W,c}(X, \mathbb{Z}(n))$

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2 Proof of Theorem I

At the heart of our constructions is an arithmetic duality theorem for cycle complexes established by Thomas Geisser in [15]. The purpose of this section is to deduce Theorem I from Geisser's duality. We would like to obtain a quasi-isomorphism of complexes

$$R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n)) \xrightarrow{\cong} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]).$$

Here $R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n))$ denotes the modified étale cohomology with compact support, described in Appendix B. We note that [15] uses the notation “ $R\Gamma_c$ ” for our “ $R\widehat{\Gamma}_c$ ”, but we take special care to distinguish the two things, since we also need the usual étale cohomology with compact support $R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n))$.

We split our proof of Theorem I into two propositions.

PROPOSITION 2.1. *For any $n < 0$ we have a quasi-isomorphism of complexes*

$$(3) \quad R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n)) \cong \varinjlim_m R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]).$$

PROOF. We unwind our definition of $\mathbb{Z}(n)$ for $n < 0$ and reduce everything to the results from [15]. Since $\mathbb{Z}(n) := \bigoplus_p \varinjlim_r j_{p!} \mu_{p^r}^{\otimes n}[-1]$, it suffices to show that for every prime p and $r \geq 1$ there is a quasi-isomorphism of complexes

$$(4) \quad R\widehat{\Gamma}_c(X_{\acute{e}t}, j_{p!} \mu_{p^r}^{\otimes n}[-1]) \cong R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c/p^r(n)), \mathbb{Q}/\mathbb{Z}[-2]),$$

and then pass to the corresponding filtered colimits.

As in Definition 1.3, here j_p denotes the canonical open immersion $j_p: X[1/p] \hookrightarrow X$. We further denote by f the structure morphism $X \rightarrow \mathrm{Spec} \mathbb{Z}$ and by f_p the structure morphism $X[1/p] \rightarrow \mathrm{Spec} \mathbb{Z}[1/p]$:

$$\begin{array}{ccc} X[1/p] & \xleftarrow{j_p} & X \\ f_p \downarrow & & \downarrow f \\ \mathrm{Spec} \mathbb{Z}[1/p] & \xleftarrow{\quad} & \mathrm{Spec} \mathbb{Z} \end{array}$$

As we are going to change the base scheme, let us write $\mathrm{Hom}_X(-, -)$ for the Hom between sheaves on $X_{\acute{e}t}$ and $\underline{\mathrm{Hom}}_X(-, -)$ for the internal Hom. Instead of $\mathrm{Hom}_{\mathrm{Spec} R}$, we will simply write Hom_R .

Applying various results from [11] and [15], we obtain a quasi-isomorphism of complexes of sheaves

$$\begin{aligned}
& R\mathbf{H}\mathbf{om}_X(j_{p!}\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}_X^c(0)) \cong \\
& \cong Rj_{p*}R\mathbf{H}\mathbf{om}_{X[1/p]}(\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}_{X[1/p]}^c(0)) \quad \text{by [15, Prop. 7.10 (c)]} \\
& \cong Rj_{p*}R\mathbf{H}\mathbf{om}_{X[1/p]}(f_p^*\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}_{X[1/p]}^c(0)) \\
& \cong Rj_{p*}Rf_p^!R\mathbf{H}\mathbf{om}_{\mathbb{Z}[1/p]}(\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}_{\mathbb{Z}[1/p]}^c(0)) \quad \text{by [15, Prop. 7.10 (c)]} \\
& \cong Rj_{p*}Rf_p^!R\mathbf{H}\mathbf{om}_{\mathbb{Z}[1/p]}(\mu_{p^r}^{\otimes n}[-1], \mathbb{G}_m[1]) \quad \text{by [15, Lemma 7.4]} \\
& \cong Rj_{p*}Rf_p^!R\mathbf{H}\mathbf{om}_{\mathbb{Z}[1/p]}(\mu_{p^r}^{\otimes n}, \mathbb{G}_m)[2] \\
& \cong Rj_{p*}Rf_p^!\mu_{p^r}^{\otimes(1-n)}[2] \\
& \cong Rj_{p*}Rf_p^!\left(\mathbb{Z}_{\mathbb{Z}[1/p]}/p^r(1-n)\right)[2] \quad \text{by [11, Thm. 1.2]} \\
& \cong Rj_{p*}Rf_p^!\mathbb{Z}_{\mathbb{Z}[1/p]}^c/p^r(n) \quad \text{by (1)} \\
& \cong Rj_{p*}\mathbb{Z}_{X[1/p]}^c/p^r(n) \quad \text{by [15, Prop. 7.10 (a)]} \\
& \cong Rj_{p*}j_p^*\mathbb{Z}_X^c/p^r(n) \cong \mathbb{Z}_X^c/p^r(n) \quad \text{by [15, Thm. 7.2 (a), Prop. 2.3]}
\end{aligned}$$

After applying $R\Gamma(X_{\acute{e}t}, -)$, we get a quasi-isomorphism of complexes of abelian groups

$$R\mathbf{H}\mathbf{om}(j_{p!}\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}_X^c(0)) \cong R\Gamma(X_{\acute{e}t}, \mathbb{Z}_X^c/p^r(n)).$$

Now according to the duality [15, Theorem 7.8],

$$R\mathbf{H}\mathbf{om}(j_{p!}\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c(0)) \cong R\mathbf{H}\mathbf{om}(R\widehat{\Gamma}_c(X_{\acute{e}t}, j_{p!}\mu_{p^r}^{\otimes n}[-1]), \mathbb{Q}/\mathbb{Z}[-2]).$$

What we end up with is a quasi-isomorphism

$$R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c/p^r(n)) \cong R\mathbf{H}\mathbf{om}(R\widehat{\Gamma}_c(X_{\acute{e}t}, j_{p!}\mu_{p^r}^{\otimes n}[-1]), \mathbb{Q}/\mathbb{Z}[-2]).$$

The groups $\widehat{H}_c^i(X_{\acute{e}t}, j_{p!}\mu_{p^r}^{\otimes n}[-1])$ are finite (the sheaves $j_{p!}\mu_{p^r}^{\otimes n}$ are constructible), so applying $R\mathbf{H}\mathbf{om}(-, \mathbb{Q}/\mathbb{Z}[-2])$ yields (4). \square

To conclude the proof of Theorem I, we identify the complex on the right-hand side of (3). For this, we need Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$.

PROPOSITION 2.2. *Assuming Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, there is a quasi-isomorphism*

$$\varinjlim_m R\mathbf{H}\mathbf{om}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]) \cong R\mathbf{H}\mathbf{om}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]).$$

PROOF. Consider short exact sequences

$$0 \rightarrow H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))_m \rightarrow H^i(X_{\acute{e}t}, \mathbb{Z}/m\mathbb{Z}^c(n)) \rightarrow {}_m H^{i+1}(X_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow 0$$

If we now take $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ and filtered colimits \varinjlim_m , we get

$$(5) \quad 0 \rightarrow \varinjlim_m \mathrm{Hom}({}_m H^{i+1}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \rightarrow \\ \varinjlim_m \mathrm{Hom}(H^i(X_{\acute{e}t}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \rightarrow \\ \varinjlim_m \mathrm{Hom}(H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))_m, \mathbb{Q}/\mathbb{Z}) \rightarrow 0$$

By Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, the group $H^{i+1}(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is finitely generated, and hence the first \varinjlim_m in the short exact sequence (5) vanishes, and we obtain isomorphisms

$$\varinjlim_m \mathrm{Hom}(H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))_m, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \varinjlim_m \mathrm{Hom}(H^i(X_{\acute{e}t}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}).$$

It remains to note that the left-hand side is canonically isomorphic to $\mathrm{Hom}(H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z})$, again thanks to the finite generation of $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$, under Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$.

To see this, observe that if A is a finitely generated abelian group, there is a canonical isomorphism

$$\varinjlim_m \mathrm{Hom}(A_m, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$$

induced by $A \rightarrow A_m$, and then applying the functor $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ and \varinjlim_m . Since \mathbb{Q}/\mathbb{Z} is a torsion group, any homomorphism $A \rightarrow \mathbb{Q}/\mathbb{Z}$ is killed by some m , hence factors through A_m . \square

3 $G_{\mathbb{R}}$ -equivariant cohomology of $X(\mathbb{C})$

We begin with some elementary homological algebra.

LEMMA 3.1. *Let A^\bullet be a perfect complex of $\mathbb{Z}G_{\mathbb{R}}$ -modules.*

- 1) *The complex $A^\bullet \otimes^{\mathbf{L}} \mathbb{Q}/\mathbb{Z}$ is of cofinite type.*
- 2) *$R\Gamma(G_{\mathbb{R}}, A^\bullet \otimes \mathbb{Q}) \cong (A^\bullet \otimes \mathbb{Q})^{G_{\mathbb{R}}}$ is a perfect complex of \mathbb{Q} -vector spaces, and the complex $R\widehat{\Gamma}(G_{\mathbb{R}}, A^\bullet \otimes \mathbb{Q})$ is quasi-isomorphic to 0.*
- 3) *$R\widehat{\Gamma}(G_{\mathbb{R}}, A^\bullet \otimes^{\mathbf{L}} \mathbb{Q}/\mathbb{Z}) \cong R\widehat{\Gamma}(G_{\mathbb{R}}, A^\bullet[+1])$, and these complexes have finite 2-torsion cohomology.*
- 4) *$R\Gamma(G_{\mathbb{R}}, A^\bullet)$ is almost perfect, and $R\Gamma(G_{\mathbb{R}}, A^\bullet \otimes^{\mathbf{L}} \mathbb{Q}/\mathbb{Z})$ is almost of cofinite type.*

PROOF. The universal coefficient theorem gives us short exact sequences

$$0 \rightarrow H^i(A^\bullet)_m \rightarrow H^i(A^\bullet \otimes^{\mathbf{L}} \mathbb{Z}/m\mathbb{Z}) \rightarrow {}_m H^{i+1}(A^\bullet) \rightarrow 0$$

The colimit of these over m is

$$0 \rightarrow H^i(A^\bullet) \otimes \mathbb{Q}/\mathbb{Z} \rightarrow H^i(A^\bullet \otimes^{\mathbf{L}} \mathbb{Q}/\mathbb{Z}) \rightarrow H^{i+1}(A^\bullet)_{\text{tor}} \rightarrow 0$$

Here $H^i(A^\bullet) \otimes \mathbb{Q}/\mathbb{Z}$ is injective, hence the short exact sequence splits. We see that $H^i(A^\bullet \otimes^{\mathbf{L}} \mathbb{Q}/\mathbb{Z})$ is of cofinite type and vanishes for $|i| \gg 0$, i.e. that $A^\bullet \otimes^{\mathbf{L}} \mathbb{Q}/\mathbb{Z}$ is of cofinite type.

Let us now consider the spectral sequences

$$(6) \quad E_2^{pq} = H^p(G_{\mathbb{R}}, H^q(A^\bullet \otimes \mathbb{Q})) \implies H^{p+q}(G_{\mathbb{R}}, A^\bullet \otimes \mathbb{Q}),$$

$$(7) \quad E_2^{pq} = \widehat{H}^p(G_{\mathbb{R}}, H^q(A^\bullet \otimes \mathbb{Q})) \implies \widehat{H}^{p+q}(G_{\mathbb{R}}, A^\bullet \otimes \mathbb{Q}).$$

We recall that $H^p(G_{\mathbb{R}}, -)$ are 2-torsion groups for $p > 0$. Since $H^q(A^\bullet \otimes \mathbb{Q})$ are \mathbb{Q} -vector spaces, it follows that $E_2^{pq} = 0$ for $p > 0$ in (6), and the spectral sequence degenerates. Similarly, the Tate cohomology groups $\widehat{H}^p(G_{\mathbb{R}}, H^q(A^\bullet \otimes \mathbb{Q}))$ are trivial for *all* p for the same reasons, so that (7) is trivial. This proves part 2).

Part 3) now follows from the distinguished triangle

$$R\widehat{\Gamma}(G_{\mathbb{R}}, A^\bullet) \rightarrow R\widehat{\Gamma}(G_{\mathbb{R}}, A^\bullet \otimes \mathbb{Q}) \rightarrow R\widehat{\Gamma}(G_{\mathbb{R}}, A^\bullet \otimes^{\mathbf{L}} \mathbb{Q}/\mathbb{Z}) \rightarrow R\widehat{\Gamma}(G_{\mathbb{R}}, A^\bullet)[1]$$

Next, examining the spectral sequence

$$E_2^{pq} = H^p(G_{\mathbb{R}}, H^q(A^\bullet)) \implies H^{p+q}(G_{\mathbb{R}}, A^\bullet),$$

we see that the groups $H^i(G_{\mathbb{R}}, A^\bullet)$ are finitely generated, zero for $i \ll 0$, and torsion for $i \gg 0$. The latter is 2-torsion. To see that, let $P_\bullet \rightarrow \mathbb{Z}$ be the bar-resolution of \mathbb{Z} by free $\mathbb{Z}G_{\mathbb{R}}$ -modules. Consider the morphism of complexes

$$\begin{array}{cccccccc} \cdots & \longrightarrow & P_3 & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 \\ & & \downarrow 2 & & \downarrow 2 & & \downarrow 2 & & \downarrow 2-N & & \\ \cdots & \longrightarrow & P_3 & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 \end{array}$$

where N denotes the norm map. The proof of [41, Theorem 6.5.8] shows that the above morphism induces multiplication by 2 on $H^i(G_{\mathbb{R}}, -)$ for $i > 0$, and it is null-homotopic. Since A^\bullet is bounded, we see that the above morphism induces multiplication by 2 on $H^i(G_{\mathbb{R}}, A^\bullet)$ for $i \gg 0$.

Similarly, analyzing

$$E_2^{pq} = H^p(G_{\mathbb{R}}, H^q(A^\bullet \otimes^{\mathbf{L}} \mathbb{Q}/\mathbb{Z})) \implies H^{p+q}(G_{\mathbb{R}}, A^\bullet \otimes^{\mathbf{L}} \mathbb{Q}/\mathbb{Z}).$$

we see that $H^i(G_{\mathbb{R}}, A^{\bullet} \otimes^{\mathbf{L}} \mathbb{Q}/\mathbb{Z})$ are groups of cofinite type. To see that these are finite 2-torsion for $i \gg 0$, consider the triangle

$$R\Gamma(G_{\mathbb{R}}, A^{\bullet}) \rightarrow R\Gamma(G_{\mathbb{R}}, A^{\bullet} \otimes \mathbb{Q}) \rightarrow R\Gamma(G_{\mathbb{R}}, A^{\bullet} \otimes^{\mathbf{L}} \mathbb{Q}/\mathbb{Z}) \rightarrow R\Gamma(G_{\mathbb{R}}, A^{\bullet})[1]$$

Here $R\Gamma(G_{\mathbb{R}}, A^{\bullet} \otimes \mathbb{Q})$ is bounded, and therefore $H^i(G_{\mathbb{R}}, A^{\bullet} \otimes^{\mathbf{L}} \mathbb{Q}/\mathbb{Z}) \cong H^{i+1}(G_{\mathbb{R}}, A^{\bullet})$ for $i \gg 0$. \square

PROPOSITION 3.2. *Let X be an arithmetic scheme. Then $X(\mathbb{C})$ has the following types of complexes as its cohomology:*

	$A = \mathbb{Z}$	$A = \mathbb{Q}$	$A = \mathbb{Q}/\mathbb{Z}$
$R\Gamma_c(X(\mathbb{C}), A(n))$	<i>perfect</i> _{/\mathbb{Z}}	<i>perfect</i> _{/\mathbb{Q}}	<i>cofinite type</i>
$R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), A(n))$	<i>almost perfect</i>	<i>perfect</i> _{/\mathbb{Q}}	<i>almost cofinite type</i>
$R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), A(n))$	<i>finite 2-torsion</i>	$\cong 0$	<i>finite 2-torsion</i>

Moreover, there is an isomorphism

$$(8) \quad \widehat{H}_c^i(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \cong H_c^i(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \quad \text{for } i \geq 2 \dim X - 1.$$

PROOF. The perfectness of $R\Gamma_c(X(\mathbb{C}), \mathbb{Z}(n))$ follows from the fact that $X(\mathbb{C})$ has the homotopy type of a finite CW-complex. This result goes back to van der Waerden [40]; more recent expositions (of more general results) can be found e.g. in [29] and [22]. The rest of the table is an application of the previous lemma to $R\Gamma_c(X(\mathbb{C}), \mathbb{Z}(n))$.

Finally, for (8), consider the spectral sequences

$$\begin{aligned} \widehat{E}_2^{pq} &= \widehat{H}^p(G_{\mathbb{R}}, H_c^q(X(\mathbb{C}), \mathbb{Z}(n))) \implies \widehat{H}_c^i(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)), \\ E_2^{pq} &= H^p(G_{\mathbb{R}}, H_c^q(X(\mathbb{C}), \mathbb{Z}(n))) \implies H_c^i(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)). \end{aligned}$$

Here $\widehat{H}^p(G_{\mathbb{R}}, -) \cong H^p(G_{\mathbb{R}}, -)$ for $p \geq 1$. Moreover, $H_c^q(X(\mathbb{C}), \mathbb{Z}(n)) = 0$ for $q \geq 2 \dim X - 1$, for the reasons of topological dimension of $X(\mathbb{C})$. \square

4 Some consequences of Theorem I

Now we deduce some consequences from the duality Theorem I.

LEMMA 4.1. *The canonical morphism $\phi^i: \widehat{H}_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) \rightarrow H_c^i(X_{\acute{e}t}, \mathbb{Z}(n))$ sits in a long exact sequence*

$$\begin{aligned} \cdots \rightarrow \widehat{H}_c^{i-1}(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \rightarrow \widehat{H}_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) \xrightarrow{\phi^i} H_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) \\ \rightarrow \widehat{H}_c^i(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \rightarrow \cdots \end{aligned}$$

where the groups $\widehat{H}_c^i(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))$ are finite 2-torsion. In particular,

- 1) the kernel and cokernel of ϕ^i are finite 2-torsion,
- 2) if $X(\mathbb{R}) = \emptyset$, then $R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) = 0$ and $\widehat{H}_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) \cong H_c^i(X_{\acute{e}t}, \mathbb{Z}(n))$.

PROOF. The exact sequence follows from the definition of modified étale cohomology with compact support and Artin's comparison theorem. This is proved in [9, Lemma 6.14]. In particular, the argument shows that $R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \cong R\widehat{\Gamma}(G_{\mathbb{R}}, v^*Rf_*\mathbb{Z}(n))$ where $v: \text{Spec } \mathbb{C} \rightarrow \text{Spec } \mathbb{Z}$ and $f: X \rightarrow \text{Spec } \mathbb{Z}$, and $R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) = 0$ if $X(\mathbb{R}) = \emptyset$.

The fact that $\widehat{H}_c^i(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))$ are finite 2-torsion is a part of Proposition 3.2. \square

PROPOSITION 4.2. *Let X be an arithmetic scheme of dimension d satisfying Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ for $n < 0$.*

- 1) If $X(\mathbb{R}) = \emptyset$, then $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) = 0$ for $i > 1$ or $i < -2d$.
- 2) In general, $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) = 0$ for $i < -2d$, and $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is a finite 2-torsion group for $i > 1$.
- 3) If X/\mathbb{F}_q is a variety over a finite field, then the groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are finite for all $i \in \mathbb{Z}$.

In general, we have the following cohomology:

groups	type	$i \ll 0$	$i \gg 0$
$H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$	finitely generated	0 for $i < -2d$	finite 2-torsion for $i > 1$
$\widehat{H}_c^i(X_{\acute{e}t}, \mathbb{Z}(n))$	cofinite	finite 2-torsion for $i < 1$	0 for $i > 2d + 2$
$H_c^i(X_{\acute{e}t}, \mathbb{Z}(n))$	cofinite	0 for $i < 1$	finite 2-torsion for $i > 2d + 2$

In particular, $R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is an almost perfect complex, while $R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n))$ is almost of cofinite type in the sense of Definition 1.1.

PROOF. If $X(\mathbb{R}) = \emptyset$, then our duality Theorem I gives

$$\text{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \cong \widehat{H}_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) \stackrel{X(\mathbb{R})=\emptyset}{\cong} H_c^i(X_{\acute{e}t}, \mathbb{Z}(n)).$$

We have $H_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) = 0$ for $i < 1$ by the definition of $\mathbb{Z}(n)$, and $H_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) = H^{i-1}(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}(n)) = 0$ for $i > 2d + 2$ for the reasons of ℓ -adic cohomological dimension [1, Exposé X, Théorème 6.2]. This proves part 1) of the proposition.

In part 2), the group $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is finite 2-torsion for $i > 1$, thanks to part 1) and Lemma 4.1. Moreover, we have $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) \cong H^i(X_{\acute{e}t}, \mathbb{Q}^c(n))$ for $i < -2d$ according

to [34, Lemma 5.12]. Conjecture $L^c(X_{\acute{e}t}, n)$ implies that these groups are \mathbb{Q} -vector spaces finitely generated over \mathbb{Z} , hence trivial.

In part 3), the cohomology groups $H^i(X_{\acute{e}t}, \mathbb{Z}(n)) = H^{i-1}(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}(n))$ are finite for $n < 0$ by [23, Theorem 3]. \square

REMARK 4.3. If X is proper and regular of dimension d , then using (1), we note that the Beilinson–Soulé vanishing conjecture (see, for example, [24, §4.3.4]) predicts that $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) = 0$ for $i < -2d$. Therefore, we proved this under Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$.

5 Complex $R\Gamma_{fg}(X, \mathbb{Z}(n))$

The purpose of this section is to define auxiliary complexes $R\Gamma_{fg}(X, \mathbb{Z}(n))$, which are used below in the construction of Weil-étale cohomology.

DEFINITION 5.1. Assuming Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, consider a morphism $\alpha_{X,n}$ in the derived category $\mathbf{D}(\mathbb{Z})$ given by the composition

$$\begin{array}{ccc} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & \xrightarrow{\mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}} & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]) \\ & \searrow \alpha_{X,n} & \uparrow \cong \\ & & R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n)) \\ & & \downarrow \mathrm{proj.} \\ & & R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \end{array}$$

Here the first arrow is induced by the canonical projection $\mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}$, and the last arrow is the canonical projection from the modified cohomology with compact support to the usual cohomology with compact support (see Appendix B).

We define the complex $R\Gamma_{fg}(X, \mathbb{Z}(n))$ as a cone of $\alpha_{X,n}$:

$$\begin{aligned} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & \xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \rightarrow R\Gamma_{fg}(X, \mathbb{Z}(n)) \\ & \rightarrow R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) \end{aligned}$$

Further, we denote

$$H_{fg}^i(X, \mathbb{Z}(n)) := H^i(R\Gamma_{fg}(X, \mathbb{Z}(n))).$$

REMARK 5.2. Under Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, the groups $H_c^i(X_{\acute{e}t}, \mathbb{Z}(n))$ are of cofinite type by Theorem I, while $R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2])$ is a complex of \mathbb{Q} -vector spaces. Therefore, the morphism $\alpha_{X,n}$ is completely determined by the maps between cohomology groups

$$H^i(\alpha_{X,n}): \mathrm{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) \rightarrow H_c^i(X_{\acute{e}t}, \mathbb{Z}(n))$$

—see Lemma A.5.

REMARK 5.3. We note that our $R\Gamma_{fg}(X, \mathbb{Z}(n))$ plays the same role as $R\Gamma_W(\overline{X}_{ét}, \mathbb{Z}(n))$ in [9, Definition 3.6]. We use a different notation since Flach and Morin work with the Artin–Verdier topology and their complex $R\Gamma_W(\overline{X}_{ét}, \mathbb{Z}(n))$ is perfect, while our complex can have finite 2-torsion in arbitrarily high degree.

We first note that the definition simplifies when X has no real places.

PROPOSITION 5.4. *If $X(\mathbb{R}) = \emptyset$, then*

$$R\Gamma_{fg}(X, \mathbb{Z}(n)) \cong R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Z}[-1]).$$

PROOF. In this case $R\widehat{\Gamma}_c(X_{ét}, \mathbb{Z}(n)) \rightarrow R\Gamma_c(X_{ét}, \mathbb{Z}(n))$ is the identity morphism, and therefore $\alpha_{X,n}$ sits in the following commutative diagram with distinguished columns:

$$\begin{array}{ccc} R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & \xrightarrow{\mathrm{id}} & R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \\ \downarrow \alpha_{X,n} & & \downarrow \\ R\Gamma_c(X_{ét}, \mathbb{Z}(n)) & \xrightarrow[\text{Theorem I}]{\cong} & R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]) \\ \downarrow & & \downarrow \\ R\Gamma_{fg}(X, \mathbb{Z}(n)) & \xrightarrow[\cong]{\text{-----}} & R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Z}[-1]) \\ \downarrow & & \downarrow \\ R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) & \xrightarrow{\mathrm{id}} & R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) \end{array}$$

Here the first column is our definition of $R\Gamma_{fg}(X, \mathbb{Z}(n))$, and the second column is induced by the distinguished triangle $\mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Z}[1]$. \square

PROPOSITION 5.5. *Assuming Conjecture $\mathbf{L}^c(X_{ét}, n)$, the complex $R\Gamma_{fg}(X, \mathbb{Z}(n))$ is almost perfect in the sense of Definition 1.1, i.e. its cohomology groups $H_{fg}^i(X, \mathbb{Z}(n))$ are finitely generated, trivial for $i \ll 0$, and 2-torsion for $i \gg 0$.*

PROOF. By the definition of $R\Gamma_{fg}(X, \mathbb{Z}(n))$, there are short exact sequences

$$0 \rightarrow \mathrm{coker} H^i(\alpha_{X,n}) \rightarrow H_{fg}^i(X, \mathbb{Z}(n)) \rightarrow \ker H^{i+1}(\alpha_{X,n}) \rightarrow 0$$

The morphism $\alpha_{X,n}$ is given at the level of cohomology by

$$(9) \quad H^i(\alpha_{X,n}): \mathrm{Hom}(H^{2-i}(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}) \xrightarrow{\psi^i} \mathrm{Hom}(H^{2-i}(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \widehat{H}_c^i(X_{ét}, \mathbb{Z}(n)) \xrightarrow{\phi^i} H_c^i(X_{ét}, \mathbb{Z}(n))$$

where $H^{2-i}(X_{ét}, \mathbb{Z}^c(n))$ is a finitely generated abelian group according to $\mathbf{L}^c(X_{ét}, n)$.

Here ϕ^i has a finite 2-torsion kernel according to Lemma 4.1, and we observe that if A is a finitely generated abelian group, then for a finite subgroup $T \subset \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ the

preimage under $\mathrm{Hom}(A, \mathbb{Q}) \rightarrow \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ is finitely generated. This justifies the finite generation of $\ker H^i(\alpha_{X,n})$ for all $i \in \mathbb{Z}$.

For the morphism ψ^i we have

$$\begin{aligned} \ker \psi^i &\cong \mathrm{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z}), \\ \mathrm{coker} \psi^i &\cong \mathrm{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{\mathrm{tor}}, \mathbb{Q}/\mathbb{Z}), \end{aligned}$$

and these groups are finitely generated by $\mathbf{L}^c(X_{\acute{e}t}, n)$. The composition of morphisms (9) gives an exact sequence (ignoring the isomorphism in the middle)

$$0 \rightarrow \ker \psi^i \rightarrow \ker(\phi^i \circ \psi^i) \rightarrow \ker \psi^i \rightarrow \mathrm{coker} \psi^i \rightarrow \mathrm{coker}(\phi^i \circ \psi^i) \rightarrow \mathrm{coker} \psi^i \rightarrow 0$$

For $i \ll 0$ we have $H_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) = 0$, and therefore

$$H_{fg}^i(X, \mathbb{Z}(n)) \cong \mathrm{Hom}(H^{1-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) = 0,$$

since the group $H^{1-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is finite 2-torsion for $i \ll 0$ by Proposition 4.2.

For $i \gg 0$ we have $\mathrm{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) = 0$, so that $H_{fg}^i(X, \mathbb{Z}(n)) \cong H_c^i(X_{\acute{e}t}, \mathbb{Z}(n))$, which is finite 2-torsion by Proposition 4.2. \square

PROPOSITION 5.6. *The complex $R\Gamma_{fg}(X, \mathbb{Z}(n))$ is defined up to a unique isomorphism in the derived category $\mathbf{D}(\mathbb{Z})$.*

PROOF. The complex $R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2])$ consists of \mathbb{Q} -vector spaces, and $R\Gamma_{fg}(X, \mathbb{Z}(n))$ is almost perfect, so we are in the situation of Corollary A.3. \square

PROPOSITION 5.7. *Suppose that Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ holds and consider the distinguished triangle defining $R\Gamma_{fg}(X, \mathbb{Z}(n))$:*

$$\begin{aligned} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) &\xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \xrightarrow{f} R\Gamma_{fg}(X, \mathbb{Z}(n)) \\ &\xrightarrow{g} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) \end{aligned}$$

1) *The morphism g induces an isomorphism*

$$g \otimes \mathbb{Q}: R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes \mathbb{Q} \xrightarrow{\cong} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]).$$

2) *For each $m \geq 1$ the morphism f induces an isomorphism*

$$f \otimes \mathbb{Z}/m\mathbb{Z}: R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \otimes^{\mathbf{L}} \mathbb{Z}/m\mathbb{Z} \xrightarrow{\cong} R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes^{\mathbf{L}} \mathbb{Z}/m\mathbb{Z}$$

3) *For any prime ℓ the morphism f induces an isomorphism*

$$\varprojlim_r H_c^i(X_{\acute{e}t}, \mathbb{Z}/\ell^r(n)) \cong H_{fg}^i(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_\ell.$$

PROOF. The groups $H_c^i(X_{\acute{e}t}, \mathbb{Z}(n))$ are all torsion, and therefore $R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \otimes \mathbb{Q} \cong 0$ in the derived category. Similarly, the complexes of \mathbb{Q} -vector spaces $R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[\cdots])$ are killed by tensoring with $\mathbb{Z}/m\mathbb{Z}$. This proves 1) and 2).

Now 2) implies 3): by the finite generation of $H_{fg}^i(X, \mathbb{Z}(n))$, we have

$$\varprojlim_r H_c^i(X_{\acute{e}t}, \mathbb{Z}/\ell^r(n)) \stackrel{2)}{\cong} \varprojlim_r H_{fg}^i(X, \mathbb{Z}/\ell^r(n)) \cong \varprojlim_r H_{fg}^i(X, \mathbb{Z}(n))/\ell^r \cong H_{fg}^i(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_\ell. \quad \square$$

The groups $H_{fg}^i(X, \mathbb{Z}(n))$ provide an integral model for ℓ -adic cohomology in the following sense (see also [11, §8]).

COROLLARY 5.8. *Let X be an arithmetic scheme satisfying Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ for $n < 0$. Then*

$$H_{fg}^i(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_\ell \cong H_c^i(X[1/\ell]_{\acute{e}t}, \mathbb{Z}_\ell(n)),$$

where the right-hand side denotes ℓ -adic cohomology with compact support.

PROOF. We have $\mathbb{Z}(n)/\ell^r \cong j_{\ell!} \mu_m^{\otimes n}$. Now by part 3) of the previous proposition,

$$H_{fg}^i(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_\ell \cong \varprojlim_r H_c^i(X_{\acute{e}t}, j_{\ell!} \mu_m^{\otimes n}) \cong \varprojlim_r H_c^i(X[1/\ell]_{\acute{e}t}, \mu_{\ell^r}^{\otimes n}) \stackrel{\mathrm{dfn}}{=} H_c^i(X[1/\ell]_{\acute{e}t}, \mathbb{Z}_\ell(n)). \quad \square$$

6 Proof of Theorem II

The aim of this section is to prove Theorem II. We recall that it states that the morphism of complexes u_∞^* , defined as the composition

$$\begin{array}{ccc} R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) & \overset{u_\infty^*}{\dashrightarrow} & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \\ \parallel & & \uparrow \\ R\Gamma_c(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}(n))[-1] & \xrightarrow{v_\infty^*[-1]} & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))[-1] \end{array}$$

is torsion. Here the morphism $v_\infty^* : R\Gamma_c(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}(n)) \rightarrow R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))$ is induced by the comparison functor $\alpha^* : \mathbf{Sh}(X_{\acute{e}t}) \rightarrow \mathbf{Sh}(G_{\mathbb{R}}, X(\mathbb{C}))$, as explained in Proposition B.5. We first ensure that α^* identifies the sheaf $\mathbb{Q}/\mathbb{Z}(n)$ on $X_{\acute{e}t}$ from Definition 1.3 with the $G_{\mathbb{R}}$ -equivariant sheaf $\mathbb{Q}/\mathbb{Z}(n) := \frac{(2\pi i)^n \mathbb{Q}}{(2\pi i)^n \mathbb{Z}}$ on $X(\mathbb{C})$.

PROPOSITION 6.1. *For the sheaf $\mathbb{Q}/\mathbb{Z}(n)$ on $X_{\acute{e}t}$ we have an isomorphism of $G_{\mathbb{R}}$ -equivariant constant sheaves on $X(\mathbb{C})$*

$$\alpha^* \mathbb{Q}/\mathbb{Z}(n) \cong \mathbb{Q}/\mathbb{Z}(n).$$

PROOF. We first compute that the functor α^* sends the sheaf $\mu_m^{\otimes n}$ on $X_{\acute{e}t}$ to the constant $G_{\mathbb{R}}$ -equivariant sheaf $\frac{(2\pi i)^n \mathbb{Z}}{m(2\pi i)^n \mathbb{Z}}$ on $X(\mathbb{C})$:

$$\begin{aligned} \alpha^* \mu_m^{\otimes n} &\cong \mu_m(\mathbb{C})^{\otimes n} := \underline{\mathrm{Hom}}(\mu_m(\mathbb{C})^{\otimes (-n)}, \mathbb{Z}/m\mathbb{Z}) \\ &\cong \frac{(2\pi i)^n \mathbb{Z}}{m(2\pi i)^n \mathbb{Z}} \end{aligned}$$

—here the first isomorphism comes from the definition of α^* given in Appendix B, and the second isomorphism comes from the corresponding isomorphism of $G_{\mathbb{R}}$ -modules.

Since α^* preserves colimits (Lemma B.4), we have

$$\alpha^* \mathbb{Q}/\mathbb{Z}(n) = \alpha^* \left(\bigoplus_p \varinjlim_r j_{p!} \mu_{p^r}^{\otimes n} \right) \cong \varinjlim_m \alpha^* \mu_m^{\otimes n} \cong \varinjlim_m \frac{(2\pi i)^n \mathbb{Z}}{m(2\pi i)^n \mathbb{Z}} \cong \frac{(2\pi i)^n \mathbb{Q}}{(2\pi i)^n \mathbb{Z}}. \quad \square$$

We proceed with our proof of Theorem II. This seems nontrivial; our argument (motivated by [9], where it is given for a proper and regular X) is based on the following result about ℓ -adic cohomology.

PROPOSITION 6.2. *Let X be an arithmetic scheme and $n < 0$. Then for any prime ℓ we have*

$$(H_c^i(X_{\overline{\mathbb{Q}}, \acute{e}t}, \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}(n))^{G_{\mathbb{Q}}})_{\mathrm{div}} = 0.$$

PROOF. According to the basic results on ℓ -adic cohomology [18, Exposé VI], there exists a prime $p \neq \ell$ such that

$$(10) \quad H_c^i(X_{\overline{\mathbb{Q}}, \acute{e}t}, \mathbb{Z}_{\ell}(n)) \cong H_c^i(X_{\overline{\mathbb{F}}_p, \acute{e}t}, \mathbb{Z}_{\ell}(n)).$$

We denote by I_p the inertia subgroup of the absolute Galois group $G_{\mathbb{Q}_p}$:

$$1 \rightarrow I_p \rightarrow G_{\mathbb{Q}_p} \rightarrow G_{\mathbb{F}_p} \rightarrow 1$$

The isomorphism (10) is equivariant under the $G_{\mathbb{Q}_p}$ -action on the left-hand side and $G_{\mathbb{F}_p}$ -action on the right-hand side. We have

$$H_c^i(X_{\overline{\mathbb{Q}}, \acute{e}t}, \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}(n))^{G_{\mathbb{Q}}} \twoheadrightarrow H_c^i(X_{\overline{\mathbb{Q}}, \acute{e}t}, \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}(n))^{G_{\mathbb{Q}_p}/I_p} \cong H_c^i(X_{\overline{\mathbb{F}}_p, \acute{e}t}, \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}(n))^{G_{\mathbb{F}_p}},$$

so it suffices to show that

$$(H_c^i(X_{\overline{\mathbb{F}}_p, \acute{e}t}, \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}(n))^{G_{\mathbb{F}_p}})_{\mathrm{div}} = 0.$$

The long exact sequence of $G_{\mathbb{F}_p}$ -modules

$$\begin{aligned} \cdots \rightarrow H_c^i(X_{\overline{\mathbb{F}}_p, \acute{e}t}, \mathbb{Z}_{\ell}(n)) \rightarrow H_c^i(X_{\overline{\mathbb{F}}_p, \acute{e}t}, \mathbb{Q}_{\ell}(n)) \rightarrow H_c^i(X_{\overline{\mathbb{F}}_p, \acute{e}t}, \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}(n)) \\ \rightarrow H_c^{i+1}(X_{\overline{\mathbb{F}}_p, \acute{e}t}, \mathbb{Z}_{\ell}(n)) \rightarrow \cdots \end{aligned}$$

induces short exact sequences

$$(11) \quad 0 \rightarrow H_c^i(X_{\overline{\mathbb{F}_p}, \acute{e}t}, \mathbb{Z}_\ell(n))_{cotor} \rightarrow H_c^i(X_{\overline{\mathbb{F}_p}, \acute{e}t}, \mathbb{Q}_\ell(n)) \rightarrow H_c^i(X_{\overline{\mathbb{F}_p}, \acute{e}t}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div} \rightarrow 0$$

According to [19, Exposé XXI, 5.5.3], the eigenvalues of the geometric Frobenius acting on $H_c^i(X_{\overline{\mathbb{F}_p}, \acute{e}t}, \mathbb{Q}_\ell)$ are algebraic integers. After twisting \mathbb{Q}_ℓ by n , the eigenvalues will lie in $p^{-n}\overline{\mathbb{Z}}$. Since $n < 0$ by our assumption, this implies that 1 does not appear as an eigenvalue, and hence

$$H_c^i(X_{\overline{\mathbb{F}_p}, \acute{e}t}, \mathbb{Q}_\ell(n))^{G_{\mathbb{F}_p}} = 0.$$

Thus, after taking the $G_{\mathbb{F}_p}$ -invariants in (11), we obtain

$$0 \rightarrow (H_c^i(X_{\overline{\mathbb{F}_p}, \acute{e}t}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div})^{G_{\mathbb{F}_p}} \rightarrow H^1(G_{\mathbb{F}_p}, H_c^i(X_{\overline{\mathbb{F}_p}, \acute{e}t}, \mathbb{Z}_\ell(n))_{cotor}) \rightarrow \cdots$$

This gives a monomorphism between the maximal divisible subgroups

$$((H_c^i(X_{\overline{\mathbb{F}_p}, \acute{e}t}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div})^{G_{\mathbb{F}_p}})_{div} \hookrightarrow H^1(G_{\mathbb{F}_p}, H_c^i(X_{\overline{\mathbb{F}_p}, \acute{e}t}, \mathbb{Z}_\ell(n))_{cotor})_{div}.$$

However, $H^1(G_{\mathbb{F}_p}, H_c^i(X_{\overline{\mathbb{F}_p}, \acute{e}t}, \mathbb{Z}_\ell(n))_{cotor})$ is a finitely generated \mathbb{Z}_ℓ -module, and therefore its maximal divisible subgroup is trivial. We conclude that

$$(H_c^i(X_{\overline{\mathbb{F}_p}, \acute{e}t}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{F}_p}})_{div} = ((H_c^i(X_{\overline{\mathbb{F}_p}, \acute{e}t}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div})^{G_{\mathbb{F}_p}})_{div} = 0. \quad \square$$

Proof of Theorem II. By Definition 1.4, this amounts to showing that the morphism

$$v_\infty^*: R\Gamma_c(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}(n)) \rightarrow R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))$$

is torsion. The complexes $R\Gamma_c(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}(n))$ and $R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))$ are almost of cofinite type by Proposition 4.2 and Proposition 3.2 respectively. Therefore, according to Lemma A.4, to show that $v_\infty^*: R\Gamma_c(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}(n)) \rightarrow R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))$ is torsion, it suffices to show that the corresponding morphisms on the maximal divisible subgroups

$$H_c^i(v_\infty^*)_{div}: H_c^i(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}(n))_{div} \rightarrow H_c^i(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))_{div}$$

are trivial. The morphism $H_c^i(v_\infty^*)$ factors through $H_c^i(X_{\overline{\mathbb{Q}}, \acute{e}t}, \mu^{\otimes n})^{G_{\mathbb{Q}}}$, where $\mu^{\otimes n}$ is the sheaf of all roots of unity on $X_{\overline{\mathbb{Q}}, \acute{e}t}$ twisted by n . So we have

$$\begin{array}{ccc} H_c^i(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}(n))_{div} & \xrightarrow{H_c^i(v_\infty^*)_{div}} & H_c^i(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))_{div} \\ & \searrow \text{dashed} & \nearrow \text{dashed} \\ & (H_c^i(X_{\overline{\mathbb{Q}}, \acute{e}t}, \mu^{\otimes n})^{G_{\mathbb{Q}}})_{div} & \end{array}$$

Now

$$\begin{aligned} (H_c^i(X_{\overline{\mathbb{Q}}, \acute{e}t}, \mu^{\otimes n})^{G_{\mathbb{Q}}})_{div} &\cong \left(\bigoplus_{\ell} H_c^i(X_{\overline{\mathbb{Q}}, \acute{e}t}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{Q}}} \right)_{div} \\ &\cong \bigoplus_{\ell} (H_c^i(X_{\overline{\mathbb{Q}}, \acute{e}t}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{Q}}})_{div}, \end{aligned}$$

where all the summands are trivial according by Proposition 6.2. \square

7 Weil-étale complex $R\Gamma_{W,c}(X, \mathbb{Z}(n))$

The aim of this section is to construct the Weil-étale cohomology complexes $R\Gamma_{W,c}(X, \mathbb{Z}(n))$.

LEMMA 7.1. *Let X be an arithmetic scheme and $n < 0$. Assume Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, so that the morphism $\alpha_{X,n}$ exists. Then $u_{\infty}^* \circ \alpha_{X,n} = 0$.*

$$\begin{array}{ccc} R\mathrm{Hom}(R\Gamma(X, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & & \\ \alpha_{X,n} \downarrow & \searrow =0 & \\ R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) & \xrightarrow{u_{\infty}^*} & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \end{array}$$

PROOF. The morphism $\alpha_{X,n}$ is defined on a complex of \mathbb{Q} -vector spaces, and u_{∞}^* is torsion by Theorem II. \square

DEFINITION 7.2. We let $i_{\infty}^*: R\Gamma_{fg}(X, \mathbb{Z}(n)) \rightarrow R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))$ be a morphism in $\mathbf{D}(\mathbb{Z})$ that gives a morphism of distinguished triangles

$$(12) \quad \begin{array}{ccc} R\mathrm{Hom}(R\Gamma(X, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & \longrightarrow & 0 \\ \alpha_{X,n} \downarrow & & \downarrow \\ R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) & \xrightarrow{u_{\infty}^*} & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \\ \downarrow & & \downarrow id \\ R\Gamma_{fg}(X, \mathbb{Z}(n)) & \dashrightarrow^{i_{\infty}^*} & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \\ \downarrow & & \downarrow \\ R\mathrm{Hom}(R\Gamma(X, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) & \longrightarrow & 0 \end{array}$$

In fact, this makes i_{∞}^* independent of any choices.

PROPOSITION 7.3. *There is a unique morphism i_{∞}^* that fits in the diagram (12).*

PROOF. We can apply Corollary A.3, since $R\mathrm{Hom}(R\Gamma(X, \mathbb{Z}^c(n)), \mathbb{Q}[-2])$ is a complex of \mathbb{Q} -vector spaces, and both $R\Gamma_{fg}(X, \mathbb{Z}(n))$ and $R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))$ are almost perfect by Proposition 5.5 and Proposition 3.2. \square

PROPOSITION 7.4. *The morphism i_{∞}^* is torsion.*

PROOF. Let us examine the morphism of distinguished triangles (12) that defines i_{∞}^* ; in particular, the commutative diagram

$$\begin{array}{ccc} R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X, \mathbb{Z}(n)) \\ u_{\infty}^* \downarrow & & \swarrow i_{\infty}^* \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) & & \end{array}$$

According to Corollary A.3, the morphism

$$\begin{aligned} \mathrm{Hom}_{\mathbf{D}(\mathbb{Z})}(R\Gamma_{fg}(X, \mathbb{Z}(n)), R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))) \\ \rightarrow \mathrm{Hom}_{\mathbf{D}(\mathbb{Z})}(R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)), R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))) \end{aligned}$$

induced by the composition with $R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \rightarrow R\Gamma_{fg}(X, \mathbb{Z}(n))$, is mono, and therefore

$$\begin{aligned} \mathrm{Hom}_{\mathbf{D}(\mathbb{Z})}(R\Gamma_{fg}(X, \mathbb{Z}(n)), R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))) \otimes \mathbb{Q} \rightarrow \\ \mathrm{Hom}_{\mathbf{D}(\mathbb{Z})}(R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)), R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))) \otimes \mathbb{Q} \end{aligned}$$

is also mono. However, $u_{\infty}^* \otimes \mathbb{Q} = 0$ by Theorem II, and this implies that $i_{\infty}^* \otimes \mathbb{Q} = 0$. \square

We are now ready to define the Weil-étale complexes.

DEFINITION 7.5. We let $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ be an object in the derived category $\mathbf{D}(\mathbb{Z})$ which is a mapping fiber of i_{∞}^* :

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \rightarrow R\Gamma_{fg}(X, \mathbb{Z}(n)) \xrightarrow{i_{\infty}^*} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \rightarrow R\Gamma_{W,c}(X, \mathbb{Z}(n))[1]$$

The **Weil-étale cohomology with compact support** is given by

$$H_{W,c}^i(X, \mathbb{Z}(n)) := H^i(R\Gamma_{W,c}(X, \mathbb{Z}(n))).$$

REMARK 7.6. Note that this defines $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ up to a non-unique isomorphism in $\mathbf{D}(\mathbb{Z})$, and the groups $H_{W,c}^i(X, \mathbb{Z}(n))$ are also defined up to a non-unique isomorphism. In a continuation of this paper we will make use of the determinant $\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n))$ in the sense of [25], which will be defined up to a canonical isomorphism.

However, we recall from Proposition 5.6 that $R\Gamma_{fg}(X, \mathbb{Z}(n))$ is defined up to a unique isomorphism in the derived category $\mathbf{D}(\mathbb{Z})$. If we could define $i_{\infty}^* : R\Gamma_{fg}(X, \mathbb{Z}(n)) \rightarrow R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))$ as an explicit, genuine morphism of complexes (not just as a morphism in the derived category $\mathbf{D}(\mathbb{Z})$), this would give us a canonical and functorial definition for $R\Gamma_{W,c}(X, \mathbb{Z}(n))$.

Case of varieties over finite fields

For varieties over finite fields, our Weil-étale cohomology has a simple description, and it is \mathbb{Q}/\mathbb{Z} -dual to the arithmetic homology studied by Geisser in [14].

PROPOSITION 7.7. *If X is a variety over a finite field \mathbb{F}_q , then assuming $\mathbf{L}^c(X, n)$, there is an isomorphism of complexes*

$$(13) \quad R\Gamma_{W,c}(X, \mathbb{Z}(n)) \cong R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z}[-1]),$$

and an isomorphism of finite groups

$$\begin{aligned} H_{W,c}^i(X, \mathbb{Z}(n)) &\cong \mathrm{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \\ &\cong H_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) \\ &\cong \mathrm{Hom}(H_{i-1}^c(X_{ar}, \mathbb{Z}(n)), \mathbb{Q}/\mathbb{Z}), \end{aligned}$$

where $H_{\bullet}^c(X_{ar}, \mathbb{Z}(n))$ are the arithmetic homology groups defined in [14, §3].

PROOF. Under our assumptions, $X(\mathbb{C}) = \emptyset$, and therefore $R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) = 0$, so that $R\Gamma_{W,c}(X, \mathbb{Z}(n)) \cong R\Gamma_{fg}(X, \mathbb{Z}(n))$. Finally, by Proposition 5.4, we have an isomorphism $R\Gamma_{fg}(X, \mathbb{Z}(n)) \cong R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z}[-1])$. We recall from Proposition 4.2 that the groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are finite under our assumption.

To relate this to Geisser's arithmetic homology, according to [14, Theorem 3.1], there is a long exact sequence

$$\cdots \rightarrow H_{i-1}^c(X_{\acute{e}t}, \mathbb{Z}(n)) \rightarrow H_i^c(X_{ar}, \mathbb{Z}(n)) \rightarrow CH_n(X, i-2n)_{\mathbb{Q}} \rightarrow H_{i-2}^c(X_{\acute{e}t}, \mathbb{Z}(n)) \rightarrow \cdots$$

Here the homological notation means that

$$\begin{aligned} H_i^c(X_{\acute{e}t}, \mathbb{Z}(n)) &= H^{-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \\ CH_n(X, i-2n)_{\mathbb{Q}} &= H_i^c(X_{\acute{e}t}, \mathbb{Q}(n)) = 0, \end{aligned}$$

and therefore

$$H_i^c(X_{ar}, \mathbb{Z}(n)) \cong H^{1-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)).$$

Now (13) gives

$$E_2^{p,q} = \mathrm{Ext}_{\mathbb{Z}}^p(H^{1-q}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z}) \implies H_{W,c}^{p+q}(X, \mathbb{Z}(n)),$$

and again, by finiteness of $H^{1-q}(X_{\acute{e}t}, \mathbb{Z}^c(n))$, this spectral sequence is concentrated in $p = 1$, where the interesting terms are

$$\mathrm{Ext}_{\mathbb{Z}}^1(H^{1-q}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z}) \cong \mathrm{Hom}(H^{1-q}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}),$$

so that

$$H_{W,c}^{1+i}(X, \mathbb{Z}(n)) \cong \mathrm{Hom}(H^{1-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(H_i^c(X_{ar}, \mathbb{Z}(n)), \mathbb{Q}/\mathbb{Z}). \quad \square$$

Perfectness of the complex

Our next aim is to verify that $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ is a perfect complex. From now on we tacitly assume Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$.

LEMMA 7.8. *The groups $H_{W,c}^i(X, \mathbb{Z}(n))$ are finitely generated for all $i \in \mathbb{Z}$.*

PROOF. In the long exact sequence

$$\begin{aligned} \cdots \rightarrow H_c^{i-1}(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \rightarrow H_{W,c}^i(X, \mathbb{Z}(n)) \rightarrow H_{fg}^i(X, \mathbb{Z}(n)) \\ \xrightarrow{H^i(i_{\infty}^*)} H_c^i(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \rightarrow \cdots \end{aligned}$$

the groups $H_c^i(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))$ and $H_{fg}^i(X, \mathbb{Z}(n))$ are finitely generated by Proposition 3.2, and Proposition 5.5, respectively. This implies the finite generation of $H_{W,c}^i(X, \mathbb{Z}(n))$. \square

LEMMA 7.9. *One has $H_{W,c}^i(X, \mathbb{Z}(n)) = 0$ for $i < 0$.*

PROOF. The definitions of $R\Gamma_{fg}(X, \mathbb{Z}(n))$ and $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ yield exact sequences

$$\begin{array}{ccccccc} & & H_c^{i-1}(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) & & & & \\ & & \downarrow & & & & \\ & & H_{W,c}^i(X, \mathbb{Z}(n)) & & & & \\ & & \downarrow & & & & \\ H_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) & \longrightarrow & H_{fg}^i(X, \mathbb{Z}(n)) & \longrightarrow & \text{Hom}(H^{1-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) & \longrightarrow & H_c^{i+1}(X_{\acute{e}t}, \mathbb{Z}(n)) \\ & & \downarrow & & & & \\ & & H_c^i(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) & & & & \end{array}$$

If $i < 0$, then $H_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) = H_c^i(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) = 0$. Moreover, $\text{Hom}(H^{1-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) = 0$ for $i < 0$, since $H^{1-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is finite 2-torsion (Proposition 4.2). We conclude that $H_{W,c}^i(X, \mathbb{Z}(n)) = H_{fg}^i(X, \mathbb{Z}(n)) = 0$ for $i < 0$. \square

For the vanishing of $H_{W,c}^i(X, \mathbb{Z}(n))$ for $i \gg 0$, we first establish the following auxiliary result.

LEMMA 7.10. *Let $d = \dim X$. For each prime ℓ and $i \geq 2d$ we have*

$$(14) \quad H_{W,c}^i(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_{\ell} = \widehat{H}_c^i(X[1/\ell]_{\acute{e}t}, \mathbb{Z}_{\ell}(n)),$$

where the right-hand side is defined via $\varprojlim_r \widehat{H}_c^i(X[1/\ell]_{\acute{e}t}, \mu_{\ell^r}^{\otimes n})$.

PROOF. Consider the commutative diagram with distinguished rows and columns

$$\begin{array}{ccccccc} [R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]] & \xrightarrow{\widehat{\alpha}_{X,n}} & R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n)) & \longrightarrow & R\widehat{\Gamma}_{fg}(X, \mathbb{Z}(n)) & \longrightarrow & [+1] \\ \downarrow id & & \downarrow & & \downarrow & & \downarrow id \\ [R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]] & \xrightarrow{\alpha_{X,n}} & R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X, \mathbb{Z}(n)) & \longrightarrow & [+1] \\ \downarrow & & \downarrow \widehat{u}_{\infty}^* & & \downarrow \widehat{i}_{\infty}^* & & \downarrow \\ 0 & \longrightarrow & R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) & \xrightarrow{id} & R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]] & \xrightarrow{\widehat{\alpha}_{X,n}[1]} & R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n))[1] & \longrightarrow & R\widehat{\Gamma}_{fg}(X, \mathbb{Z}(n))[1] & \longrightarrow & [+2] \end{array}$$

Here \widehat{u}_∞^* (resp. \widehat{i}_∞^*) is defined as the composition of the canonical morphism u_∞^* (resp. i_∞^*) with the projection to the Tate cohomology

$$\pi: R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \rightarrow R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)).$$

By Proposition 3.2, $H^i(\pi)$ is an isomorphism for $i \geq 2d - 1$. Therefore, the five-lemma applied to

$$\begin{array}{ccccccc} R\Gamma_{W,c}(X, \mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X, \mathbb{Z}(n)) & \xrightarrow{i_\infty^*} & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) & \longrightarrow & [+1] \\ \downarrow f & & \downarrow id & & \downarrow \pi & & \downarrow f[1] \\ R\widehat{\Gamma}_{fg}(X, \mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X, \mathbb{Z}(n)) & \xrightarrow{\widehat{i}_\infty^*} & R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) & \longrightarrow & [+1] \end{array}$$

shows that for $i \geq 2d$ holds

$$H_{W,c}^i(X, \mathbb{Z}(n)) \cong \widehat{H}_{fg}^i(X, \mathbb{Z}(n)).$$

As in Corollary 5.8, we have for a prime ℓ

$$\widehat{H}_{fg}^i(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_\ell \cong \widehat{H}_c^i(X[1/\ell]_{\acute{e}t}, \mathbb{Z}_\ell(n)). \quad \square$$

COROLLARY 7.11. *One has $H_{W,c}^i(X, \mathbb{Z}(n)) = 0$ for $i > 2d + 1$.*

PROOF. It suffices to verify that $H_{W,c}^i(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_\ell = 0$ for each prime ℓ . Thanks to the isomorphism (14), this reduces to $\widehat{H}_c^i(X[1/\ell]_{\acute{e}t}, \mathbb{Z}_\ell(n)) = 0$ for $i > 2d + 1$, which is true for the reasons of cohomological dimension [1, Exposé X, Théorème 6.2]. We note that if $\ell = 2$ and $X(\mathbb{R}) \neq \emptyset$, then the usual étale cohomology has finite 2-torsion in arbitrarily high degrees. It is important that we consider here the *modified* cohomology with compact support $\widehat{H}_c^i(-)$. To obtain the corresponding statement, combine the arguments from [1, Exposé X] with the well-known computations of modified cohomology for number fields; cf. [32, Chapter II] and [2], [31]. \square

Summarizing the above observations, we obtain the following result.

PROPOSITION 7.12. *Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ implies that $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ is a perfect complex. More precisely, $H_{W,c}^i(X, \mathbb{Z}(n))$ are finitely generated groups, and $H_{W,c}^i(X, \mathbb{Z}(n)) = 0$ for $i \notin [0, 2 \dim X + 1]$.*

Rational coefficients

PROPOSITION 7.13. *There is a non-canonical splitting*

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{Q} \cong R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q})[-1] \oplus R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}(n))[-1].$$

PROOF. The distinguished triangle defining $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ becomes after tensoring with \mathbb{Q}

$$\begin{aligned} R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{Q} &\rightarrow R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes \mathbb{Q} \xrightarrow{i_\infty^* \otimes \mathbb{Q}=0} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \otimes \mathbb{Q} \\ &\rightarrow R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{Q}[1] \end{aligned}$$

which yields a non-canonical splitting [39, Chapitre II, Corollaire 1.2.6]

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{Q} \cong R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes \mathbb{Q} \oplus R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))[-1] \otimes \mathbb{Q},$$

and we have already established in Proposition 5.7 that

$$R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes \mathbb{Q} \cong R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q})[-1]. \quad \square$$

8 Known cases of Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$

Since the main constructions of this paper assume Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, we relate it here to other known conjectures about the finite generation of étale motivic cohomology, and also describe certain schemes X for which $\mathbf{L}^c(X_{\acute{e}t}, n)$ holds unconditionally.

Flach and Morin state in [9] a slightly different conjecture $\mathbf{L}(X_{\acute{e}t}, -)$ instead of our $\mathbf{L}^c(X_{\acute{e}t}, -)$. Taking into account the relation (1) for regular schemes, we can reformulate their conjecture as follows.

CONJECTURE 8.1 ([9, Conjecture 3.2; Lemma 3.3]). *$\mathbf{L}(X_{\acute{e}t}, d-n)$: for a proper regular arithmetic scheme X and $n < 0$, the groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are finitely generated for $i \leq -2n + 1$.*

A more precise conjectural description of étale motivic cohomology is [16, Conjecture 4.12], which can be written as follows, again using (1):

CONJECTURE 8.2. *$\mathbf{L}'(X_{\acute{e}t}, d-n)$: for a proper regular arithmetic scheme X and $n < 0$, one has*

$$H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) = \begin{cases} \text{finitely generated,} & i \leq -2n, \\ \text{finite,} & i = -2n + 1, \\ \text{cofinite type,} & i \geq -2n + 2. \end{cases}$$

PROPOSITION 8.3. *Let X be a proper regular arithmetic scheme of dimension d . Then for $n < 0$*

$$\mathbf{L}^c(X_{\acute{e}t}, n) \iff \mathbf{L}(X_{\acute{e}t}, d-n) \iff \mathbf{L}'(X_{\acute{e}t}, d-n).$$

PROOF. The nontrivial implications are

$$\mathbf{L}(X_{\acute{e}t}, d - n) \implies \mathbf{L}^c(X_{\acute{e}t}, n), \quad \mathbf{L}(X_{\acute{e}t}, d - n) \implies \mathbf{L}'(X_{\acute{e}t}, d - n).$$

For the first implication, we note that by [9, Proposition 3.4], $\mathbf{L}(X_{\acute{e}t}, d - n)$ implies the Artin–Verdier duality

$$H^i(X_{\acute{e}t}, \mathbb{Z}(n)) \cong \text{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \text{ up to finite 2-torsion.}$$

Hence $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is finite 2-torsion for $i \geq 2$, and in particular for $i > -2n + 1$.

The second implication is also established in [9, Proposition 3.4]. \square

We now list some special cases where Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ is known, and therefore gives unconditional results. We follow [34, §5] very closely. For an arithmetic scheme X , we formulate the following conjecture, which is the conjunction of $\mathbf{L}^c(X_{\acute{e}t}, n)$ for all $n < 0$.

CONJECTURE 8.4. $\mathbf{L}^c(X_{\acute{e}t})$: *the cohomology groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are finitely generated for all $i \in \mathbb{Z}$ and $n < 0$.*

This is similar to [34, Definition 5.8], with the only difference that Morin also requires the finite generation of $H^i(X_{\acute{e}t}, \mathbb{Z}^c(0))$ for $i \leq 0$. Conjecture $\mathbf{L}^c(X_{\acute{e}t})$ is known for number rings, and also for certain varieties over finite fields. As in [37], [11], and [34], we consider the following class.

DEFINITION 8.5. Let $A(\mathbb{F}_q)$ be the full subcategory of the category of smooth projective varieties over a finite field \mathbb{F}_q generated by products of curves and the following operations.

- 1) If X and Y lie in $A(\mathbb{F}_q)$, then $X \sqcup Y$ lies in $A(\mathbb{F}_q)$.
- 2) If Y lies in $A(\mathbb{F}_q)$ and there are morphisms $c: X \rightarrow Y$ and $c': Y \rightarrow X$ in the category of Chow motives such that $c' \circ c: X \rightarrow X$ is a multiplication by constant, then X lies in $A(\mathbb{F}_q)$.
- 3) If $\mathbb{F}_{q^m}/\mathbb{F}_q$ is a finite extension and $X_{\mathbb{F}_{q^m}} = X \times_{\text{Spec } \mathbb{F}_q} \text{Spec } \mathbb{F}_{q^m}$ lies in $A(\mathbb{F}_{q^m})$, then X lies in $A(\mathbb{F}_q)$.
- 4) If X and Y lie in $A(\mathbb{F}_q)$, and Y is a closed subscheme of X , then the blowup of X along Y lies in $A(\mathbb{F}_q)$.

The following is similar to [34, Definition 5.9].

DEFINITION 8.6. Let $\mathcal{L}(\mathbb{Z})$ be the full subcategory of arithmetic schemes generated by the following objects:

- the empty scheme \emptyset ,
- $\text{Spec } \mathcal{O}_F$ for a number field F ,
- varieties $X \in A(\mathbb{F}_q)$ for any finite field \mathbb{F}_q ,

and the following operations.

- $\mathcal{L}1)$ Let X be an arithmetic scheme, $Z \subset X$ a closed subscheme and $U := X \setminus Z$ its open complement. If two of three schemes X, Z, U lie in $\mathcal{L}(\mathbb{Z})$, then the third also lies in $\mathcal{L}(\mathbb{Z})$.
- $\mathcal{L}2)$ A finite disjoint union $X = \coprod_{1 \leq j \leq p} X_j$ lies in $\mathcal{L}(\mathbb{Z})$ if and only if each X_j lies in $\mathcal{L}(\mathbb{Z})$.
- $\mathcal{L}3)$ If $V \rightarrow U$ is an affine bundle and U lies in $\mathcal{L}(\mathbb{Z})$, then V also lies in $\mathcal{L}(\mathbb{Z})$.
- $\mathcal{L}4)$ If $\{U_i \rightarrow X\}_{i \in I}$ is a finite surjective family of étale morphisms such that each U_{i_0, \dots, i_p} lies in $\mathcal{L}(\mathbb{Z})$, then X also lies in $\mathcal{L}(\mathbb{Z})$.

PROPOSITION 8.7. *Conjecture $\mathbf{L}^c(X_{\text{ét}})$ holds for any arithmetic scheme $X \in \mathcal{L}(\mathbb{Z})$.*

PROOF. See the argument in [34, Proposition 5.10]. \square

Finally, we consider cellular schemes, as in [34, §5.4].

DEFINITION 8.8. Let Y be a separated scheme of finite type over $\text{Spec } k$ for a field k . We say that Y **admits a cellular decomposition** if there exists a filtration of Y by reduced closed subschemes

$$Y^{\text{red}} = Y_N \supseteq Y_{N-1} \supseteq \cdots \supseteq Y_{-1} = \emptyset$$

such that $Y_i \setminus Y_{i-1} \cong \mathbb{A}_k^{r_i}$ is isomorphic to an affine space over k .

We say that Y is **geometrically cellular** if $Y_{\bar{k}} = Y \times_{\text{Spec } k} \text{Spec } \bar{k}$ admits a cellular decomposition. This is equivalent to the existence of a finite Galois extension k'/k such that $Y_{k'}$ admits a cellular decomposition.

Finally, given an S -scheme $X \rightarrow S$ that is separated and of finite type, we say that X is **geometrically cellular** if for each $s \in S$ the corresponding fiber X_s is geometrically cellular.

PROPOSITION 8.9. *Let Y be a separated scheme of finite type over $\text{Spec } \mathbb{F}_q$. If Y is geometrically cellular, then $X \in \mathcal{L}(\mathbb{Z})$, and in particular Conjecture $\mathbf{L}^c(Y_{\text{ét}})$ holds.*

If $X \rightarrow \text{Spec } \mathcal{O}_F$ is a flat, separated scheme of finite type over the ring of integers of a number field, and X is geometrically cellular, then $X \in \mathcal{L}(\mathbb{Z})$, and in particular $\mathbf{L}^c(X_{\text{ét}})$ holds.

For a proof, we refer to [34, Proposition 5.14].

9 Comparison with the complex of Flach and Morin

This paper is based on the ideas of Flach and Morin [9], who gave a similar construction of Weil-étale cohomology $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ for a *proper and regular* arithmetic scheme X , and for *any integer weight* $n \in \mathbb{Z}$. In this section, we will go through the definitions of [9], to verify the following claim.

PROPOSITION 9.1. *Let X be a proper, regular arithmetic scheme, and $n < 0$. Assume Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$. Then the Weil-étale complex $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ defined above in §7 is isomorphic to the corresponding complex defined in [9].*

From now on we tacitly assume Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, which is also equivalent to the assumptions on motivic cohomology in [9] (see Proposition 8.3). Flach and Morin consider the case of a proper and regular arithmetic scheme X of equal dimension d . In this case, we can use the isomorphism (1) to reformulate their constructions in terms of complexes $\mathbb{Z}^c(n)$.

Moreover, they work with the Artin-Verdier étale topos $\overline{X}_{\acute{e}t}$, whose definition and basic properties can be found in [9, §6]. They consider a morphism

$$\overline{\alpha}_{X,n}: R\mathrm{Hom}(R\Gamma(X, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \rightarrow R\Gamma(\overline{X}_{\acute{e}t}, \mathbb{Z}(n)),$$

defined in a similar way to our $\alpha_{X,n}$ (Definition 5.1) using a duality similar to our Theorem I.

The notation in [9] and in this paper is intentionally the same for various objects and morphisms. However, in this section we will write, for example, $\overline{\alpha}_{X,n}$ to denote the morphism of Flach and Morin, to distinguish it from our $\alpha_{X,n}$, etc. An overline indicates that the corresponding thing comes from [9] and has something to do with the Artin-Verdier étale topos.

LEMMA 9.2. *The square*

$$(15) \quad \begin{array}{ccc} R\mathrm{Hom}(R\Gamma(X, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & \xrightarrow{\overline{\alpha}_{X,n}} & R\Gamma(\overline{X}_{\acute{e}t}, \mathbb{Z}(n)) \\ \downarrow id & & \downarrow \\ R\mathrm{Hom}(R\Gamma(X, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & \xrightarrow{\alpha_{X,n}} & R\Gamma(X_{\acute{e}t}, \mathbb{Z}(n)) \end{array}$$

commutes.

PROOF. We recall from Remark 5.2 that $\alpha_{X,n}$ is determined by the maps at the level of cohomology $H^i(\alpha_{X,n})$. The same is true for $\overline{\alpha}_{X,n}$, for the same reasons. Now [9, Theorem 3.5] defines

$$\begin{aligned} H^i(\overline{\alpha}_{X,n}): \mathrm{Hom}(H^{2-i}(X, \mathbb{Z}^c(n)), \mathbb{Q}) &\xrightarrow{\cong} \mathrm{Hom}(H^{2-i}(\overline{X}_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) \rightarrow \\ &\mathrm{Hom}(H^{2-i}(\overline{X}_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \xleftarrow{\cong} H^i(\overline{X}_{\acute{e}t}, \mathbb{Z}(n)), \end{aligned}$$

where the last isomorphism is the duality [9, Corollary 6.26]. Similarly, our morphism $\alpha_{X,n}$ gives

$$H^i(\alpha_{X,n}): \mathrm{Hom}(H^{2-i}(X, \mathbb{Z}^c(n)), \mathbb{Q}) \xrightarrow{\cong} \mathrm{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) \rightarrow \\ \mathrm{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \xleftarrow{\cong} \widehat{H}_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) \rightarrow H^i(X_{\acute{e}t}, \mathbb{Z}(n)).$$

The groups $\widehat{H}_c^i(X_{\acute{e}t}, \mathbb{Z}(n))$ and $H^i(\overline{X}_{\acute{e}t}, \mathbb{Z}(n))$ are different, but the duality in terms of $H^i(\overline{X}_{\acute{e}t}, \mathbb{Z}(n))$ is induced precisely from the duality in terms of $\widehat{H}_c^i(X_{\acute{e}t}, \mathbb{Z}(n))$ (see [9, Theorem 6.24]): we have a commutative diagram

$$\begin{array}{ccc} R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}/m\mathbb{Z}(n)) & \xrightarrow{\cong} & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]) \\ \downarrow & & \downarrow \\ R\Gamma(\overline{X}_{\acute{e}t}, \mathbb{Z}/m\mathbb{Z}(n)) & \xrightarrow{\cong} & R\mathrm{Hom}(R\Gamma(\overline{X}_{\acute{e}t}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]) \end{array}$$

and the diagram

$$\begin{array}{ccc} R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n)) & \longrightarrow & R\Gamma(X_{\acute{e}t}, \mathbb{Z}(n)) \\ \downarrow & \nearrow & \\ R\Gamma(\overline{X}_{\acute{e}t}, \mathbb{Z}(n)) & & \end{array}$$

commutes as well. We see that the diagram we are interested in commutes:

$$\begin{array}{ccccc} & & H^i(\overline{\alpha}_{X,n}) & & \\ & \searrow & \curvearrowright & \searrow & \\ \mathrm{Hom}(H^{2-i}(X, \mathbb{Z}^c(n)), \mathbb{Q}) & \longrightarrow & H^{2-i}(\overline{X}_{\acute{e}t}, \mathbb{Z}^c(n))^D & \xleftarrow{\cong} & H^i(\overline{X}_{\acute{e}t}, \mathbb{Z}(n)) \\ \downarrow \mathrm{id} & & \uparrow \text{---} & & \downarrow \\ \mathrm{Hom}(H^{2-i}(X, \mathbb{Z}^c(n)), \mathbb{Q}) & \longrightarrow & H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))^D & \xleftarrow{\cong} \widehat{H}_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) \longrightarrow & H^i(X_{\acute{e}t}, \mathbb{Z}(n)) \\ & \searrow & \curvearrowright & \searrow & \\ & & H^i(\alpha_{X,n}) & & \end{array}$$

For brevity, $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ is denoted here by A^D . □

Taking the cones of $\overline{\alpha}_{X,n}$ and $\alpha_{X,n}$, we obtain respectively the complex $R\Gamma_W(\overline{X}, \mathbb{Z}(n))$ of Flach and Morin [9, Definition 3.6] and our complex $R\Gamma_{fg}(X, \mathbb{Z}(n))$ (Definition 5.1 above).

The square (15) induces the following diagram with distinguished rows and columns

(cf. [35, Proposition 1.4.6]):

(16)

$$\begin{array}{ccccccc}
[R\Gamma(X, \mathbb{Z}^c(n)), \mathbb{Q}[-2]] & \xrightarrow{\bar{\alpha}_{X,n}} & R\Gamma(\bar{X}_{\acute{e}t}, \mathbb{Z}(n)) & \xrightarrow{f} & R\Gamma_W(\bar{X}, \mathbb{Z}(n)) & \longrightarrow & [-1] \\
\downarrow id & & \downarrow & & \downarrow \text{---} & & \downarrow id \\
[R\Gamma(X, \mathbb{Z}^c(n)), \mathbb{Q}[-2]] & \xrightarrow{\alpha_{X,n}} & R\Gamma(X_{\acute{e}t}, \mathbb{Z}(n)) & \xrightarrow{g} & R\Gamma_{fg}(X, \mathbb{Z}(n)) & \longrightarrow & [-1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & R\Gamma(X(\mathbb{R}), \tau_{\geq n+1} R\hat{\pi}_* \mathbb{Z}(n)) & \xrightarrow{id} & R\Gamma(X(\mathbb{R}), \tau_{\geq n+1} R\hat{\pi}_* \mathbb{Z}(n)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
[R\Gamma(X, \mathbb{Z}^c(n)), \mathbb{Q}[-1]] & \longrightarrow & R\Gamma(\bar{X}_{\acute{e}t}, \mathbb{Z}(n))[1] & \xrightarrow{f[1]} & R\Gamma_W(\bar{X}, \mathbb{Z}(n))[1] & \longrightarrow & [0]
\end{array}$$

Then [9, Definition 3.23] considers a morphism \bar{u}_∞^* defined via

(17)

$$\begin{array}{ccccccc}
R\Gamma(\bar{X}_{\acute{e}t}, \mathbb{Z}(n)) & \longrightarrow & R\Gamma(X_{\acute{e}t}, \mathbb{Z}(n)) & \longrightarrow & R\Gamma(X(\mathbb{R}), \tau_{\geq n+1} R\hat{\pi}_* \mathbb{Z}(n)) & \rightarrow & [+1] \\
\exists \downarrow \bar{u}_\infty^* & & \downarrow u_\infty^* & & \downarrow id & & \downarrow \bar{u}_\infty^*[1] \\
R\Gamma_W(X_\infty, \mathbb{Z}(n)) & \rightarrow & R\Gamma(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) & \rightarrow & R\Gamma(X(\mathbb{R}), \tau_{\geq n+1} R\hat{\pi}_* \mathbb{Z}(n)) & \rightarrow & [+1]
\end{array}$$

Here the complex $R\Gamma_W(X_\infty, \mathbb{Z}(n))$ is *defined* via the bottom triangle.

Then [9, Proposition 3.24] and our Proposition 7.3 above establish the existence and uniqueness of morphisms \bar{t}_∞^* and i_∞^* which make the triangles below commutative, and then the Weil-étale complexes are defined as mapping fibers of \bar{t}_∞^* and i_∞^* :

$$\begin{array}{ccc}
R\Gamma_{W,c}(\bar{X}, \mathbb{Z}(n)) & & R\Gamma_{W,c}(X, \mathbb{Z}(n)) \\
\downarrow & & \downarrow \\
R\Gamma_W(\bar{X}, \mathbb{Z}(n)) \xleftarrow{f} R\Gamma(\bar{X}_{\acute{e}t}, \mathbb{Z}(n)) & & R\Gamma_{fg}(X, \mathbb{Z}(n)) \xleftarrow{g} R\Gamma(X_{\acute{e}t}, \mathbb{Z}(n)) \\
\downarrow \bar{t}_\infty^* \swarrow \bar{u}_\infty^* & & \downarrow i_\infty^* \swarrow u_\infty^* \\
R\Gamma_W(X_\infty, \mathbb{Z}(n)) & & R\Gamma(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \\
\downarrow & & \downarrow \\
R\Gamma_{W,c}(\bar{X}, \mathbb{Z}(n))[1] & & R\Gamma_{W,c}(X, \mathbb{Z}(n))[1]
\end{array}$$

In order to compare the two resulting complexes, we note that \bar{u}_∞^* is only defined via (17), so in the diagram below from Figure 1, we can first choose \bar{t}_∞^* such that the front face gives a morphism of triangles. Then we can *declare* \bar{u}_∞^* to be the composition $\bar{t}_\infty^* \circ f$. In this way everything commutes, and we see that $R\Gamma_{W,c}(\bar{X}, \mathbb{Z}(n)) \cong R\Gamma_{W,c}(X, \mathbb{Z}(n))$.

This concludes the proof of Proposition 9.1. \square

$$\begin{array}{ccccccc}
 R\Gamma_{W,c}(\overline{X}, \mathbb{Z}(n)) & \overset{\cong}{\dashrightarrow} & R\Gamma_{W,c}(X, \mathbb{Z}(n)) & \longrightarrow & 0 & \longrightarrow & [+1] \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 R\Gamma_{W,c}(\overline{X}, \mathbb{Z}(n)) & \xrightarrow{f} & R\Gamma(\overline{X}_{\acute{e}t}, \mathbb{Z}(n)) & \longrightarrow & R\Gamma(X_{\acute{e}t}, \mathbb{Z}(n)) & \longrightarrow & R\Gamma(X(\mathbb{R}), \tau_{\geq n+1} R\widehat{\pi}_* \mathbb{Z}(n)) \longrightarrow [+1] \\
 \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\
 R\Gamma_W(\overline{X}, \mathbb{Z}(n)) & \xrightarrow{f} & R\Gamma_{fg}(X, \mathbb{Z}(n)) & \longrightarrow & R\Gamma(X(\mathbb{R}), \tau_{\geq n+1} R\widehat{\pi}_* \mathbb{Z}(n)) & \longrightarrow & [+1] \\
 \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\
 R\Gamma_W(\overline{X}, \mathbb{Z}(n)) & \xrightarrow{\bar{u}_\infty^*} & R\Gamma_{fg}(X, \mathbb{Z}(n)) & \longrightarrow & R\Gamma(X(\mathbb{R}), \tau_{\geq n+1} R\widehat{\pi}_* \mathbb{Z}(n)) & \longrightarrow & [+1] \\
 \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\
 R\Gamma_W(X_\infty, \mathbb{Z}(n)) & \xrightarrow{i_\infty^*} & R\Gamma(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) & \longrightarrow & R\Gamma(X(\mathbb{R}), \tau_{\geq n+1} R\widehat{\pi}_* \mathbb{Z}(n)) & \longrightarrow & [+1] \\
 \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\
 R\Gamma_{W,c}(\overline{X}, \mathbb{Z}(n))[1] & \overset{\cong}{\dashrightarrow} & R\Gamma_{W,c}(X, \mathbb{Z}(n))[1] & \longrightarrow & 0 & \longrightarrow & [+2]
 \end{array}$$

Figure 1: Comparison of the Weil-étale complexes from [9] and this paper, denoted $R\Gamma_{W,c}(\overline{X}, \mathbb{Z}(n))$ and $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ respectively. The top face of the prism comes from (16). The arrow \bar{t}_∞^* is chosen so that the front face is commutative. Then set $\bar{u}_\infty^* = \bar{t}_\infty^* \circ f$ so that the back face is commutative and corresponds to (17).

A Some homological algebra

This appendix contains some basic results about the derived category of abelian groups $\mathbf{D}(\mathbb{Z})$ which are used throughout the text. The following lemmas are isolated from the proofs in [9], with some modifications to treat the 2-torsion.

First, recall that every complex of abelian groups A^\bullet (not necessarily bounded) is quasi-isomorphic to its cohomology:

$$\begin{aligned} A^\bullet &\cong \coprod_{i \in \mathbb{Z}} H^i(A^\bullet)[-i] \cong \prod_{i \in \mathbb{Z}} H^i(A^\bullet)[-i] \\ &= \left(\cdots \rightarrow H^{i-1}(A^\bullet) \xrightarrow{0} H^i(A^\bullet) \xrightarrow{0} H^{i+1}(A^\bullet) \rightarrow \cdots \right). \end{aligned}$$

Here \coprod and \prod denote the coproduct and product in the category of complexes, which coincide in this case. This gives us a useful expression for morphisms in the derived category: since $\mathrm{Hom}_{\mathbf{D}(\mathbb{Z})}(A, B[i]) \cong \mathrm{Ext}_{\mathbb{Z}}^i(A, B)$, and $\mathrm{Ext}_{\mathbb{Z}}^i(A, B) = 0$ for $i > 1$, we obtain

$$\begin{aligned} \mathrm{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet, B^\bullet) &\cong \mathrm{Hom}_{\mathbf{D}(\mathbb{Z})}\left(\prod_{i \in \mathbb{Z}} H^i(A^\bullet)[-i], \prod_{j \in \mathbb{Z}} H^j(B^\bullet)[-j]\right) \\ &\cong \prod_{i \in \mathbb{Z}} \prod_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathbf{D}(\mathbb{Z})}(H^i(A^\bullet), H^j(B^\bullet)[i-j]) \\ &\cong \prod_{i \in \mathbb{Z}} \left(\mathrm{Hom}(H^i(A^\bullet), H^i(B^\bullet)) \oplus \mathrm{Ext}(H^i(A^\bullet), H^{i-1}(B^\bullet)) \right) \\ (18) \quad &\cong \prod_{i \in \mathbb{Z}} \mathrm{Hom}(H^i(A^\bullet), H^i(B^\bullet)) \oplus \prod_{i \in \mathbb{Z}} \mathrm{Ext}(H^i(A^\bullet), H^{i-1}(B^\bullet)). \end{aligned}$$

LEMMA A.1.

- 1) If C^\bullet and C'^\bullet are almost perfect in the sense of Definition 1.1, then the group $\mathrm{Hom}_{\mathbf{D}(\mathbb{Z})}(C^\bullet, C'^\bullet)$ has no nontrivial divisible subgroups.
- 2) If A^\bullet is a complex such that $H^i(A^\bullet)$ are finite-dimensional \mathbb{Q} -vector spaces and C^\bullet is a complex such that $H^i(C^\bullet)$ are finitely generated abelian groups, then the group $\mathrm{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet, C^\bullet)$ is divisible.

PROOF. In 1), we consider the decomposition (18) for $\mathrm{Hom}_{\mathbf{D}(\mathbb{Z})}(C^\bullet, C'^\bullet)$, and observe that under our assumptions, both groups

$$\prod_{i \in \mathbb{Z}} \mathrm{Hom}(H^i(C^\bullet), H^i(C'^\bullet)) \quad \text{and} \quad \prod_{i \in \mathbb{Z}} \mathrm{Ext}(H^i(C^\bullet), H^{i-1}(C'^\bullet))$$

are of the form $G \oplus T$, where G is a finitely generated abelian group and T is 2-torsion. From this we see that if $x \in \mathrm{Hom}_{\mathbf{D}(\mathbb{Z})}(C^\bullet, C'^\bullet)$ is divisible by all powers of 2, then $x = 0$.

Similarly, in part 2), we consider the decomposition (18) for $\mathrm{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet, C^\bullet)$. Under our assumptions, $\mathrm{Hom}(H^i(A^\bullet), H^i(C^\bullet)) = 0$ for all i , and each $\mathrm{Ext}(H^i(A^\bullet), H^{i-1}(C^\bullet))$ is a direct sum of finitely many groups isomorphic to $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$, which is divisible. Therefore, $\mathrm{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet, C^\bullet)$ is a direct product of divisible groups, hence divisible. \square

Recall that Verdier's axiom (TR1) states that every morphism $v: A^\bullet \rightarrow B^\bullet$ can be completed to a distinguished triangle $A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \xrightarrow{w} A^\bullet[1]$. Axiom (TR3) states that for every commutative diagram with distinguished rows

$$(19) \quad \begin{array}{ccccccc} A^\bullet & \xrightarrow{u} & B^\bullet & \xrightarrow{v} & C^\bullet & \xrightarrow{w} & A^\bullet[1] \\ \downarrow f & & \downarrow g & & & & \\ A'^\bullet & \xrightarrow{u'} & B'^\bullet & \xrightarrow{v'} & C'^\bullet & \xrightarrow{w'} & A'^\bullet[1] \end{array}$$

there exists some $h: C^\bullet \rightarrow C'^\bullet$, which gives a morphism of distinguished triangles

$$(20) \quad \begin{array}{ccccccc} A^\bullet & \xrightarrow{u} & B^\bullet & \xrightarrow{v} & C^\bullet & \xrightarrow{w} & A^\bullet[1] \\ \downarrow f & & \downarrow g & & \downarrow \exists h & & \downarrow f[1] \\ A'^\bullet & \xrightarrow{u'} & B'^\bullet & \xrightarrow{v'} & C'^\bullet & \xrightarrow{w'} & A'^\bullet[1] \end{array}$$

The cone C^\bullet in (TR1) and the morphism h in (TR3) are neither unique nor canonical. Two different cones of the same morphism are necessarily isomorphic, but the isomorphism between them is not unique, because it is provided by (TR3). Let us recall a useful argument showing that things are well-defined in some special cases.

LEMMA A.2 (\approx [3, Proposition 1.1.9, Corollaire 1.1.10]). *Consider the derived category $\mathbf{D}(\mathcal{A})$ of an abelian category \mathcal{A} .*

- 1) *For a commutative diagram (19), assume that the homomorphism of abelian groups*

$$w^*: \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A^\bullet[1], C'^\bullet) \rightarrow \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(C^\bullet, C'^\bullet)$$

induced by w is trivial. Then there exists a unique morphism $h: C^\bullet \rightarrow C'^\bullet$ that gives a morphism of triangles (20).

- 2) *For a distinguished triangle $A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \xrightarrow{w} A^\bullet[1]$, assume that for any other cone C'^\bullet of u the morphism w^* is trivial. Then the cone of u is unique up to a unique isomorphism.*

PROOF. In 1), applying $\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(-, C'^\bullet)$ to the first distinguished triangle, we obtain an exact sequence of abelian groups

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A^\bullet[1], C'^\bullet) \xrightarrow{w^*} \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(C^\bullet, C'^\bullet) \xrightarrow{v^*} \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(B^\bullet, C'^\bullet).$$

If $w^* = 0$, we conclude that v^* is a monomorphism. This implies that there is a unique morphism h such that $h \circ v = v' \circ g$. Now in 2), if C^\bullet and C'^\bullet are two different cones of u , we have a commutative diagram

$$\begin{array}{ccccccc} A^\bullet & \xrightarrow{u} & B^\bullet & \xrightarrow{v} & C^\bullet & \xrightarrow{w} & A^\bullet[1] \\ \downarrow id & & \downarrow id & & \downarrow \text{dashed} & & \downarrow id \\ A^\bullet & \xrightarrow{u'} & B^\bullet & \xrightarrow{v'} & C'^\bullet & \xrightarrow{w'} & A^\bullet[1] \end{array}$$

By the triangulated five-lemma, the dashed arrow is an isomorphism, and it is unique thanks to part 1). \square

Here is a special case that we need.

COROLLARY A.3. *Consider the derived category $\mathbf{D}(\mathbb{Z})$.*

1) *Suppose we have a commutative diagram with distinguished rows (19), where A^\bullet is a complex such that $H^i(A^\bullet)$ are finite-dimensional \mathbb{Q} -vector spaces and C^\bullet, C'^\bullet are almost perfect complexes in the sense of Definition 1.1. Then there exists a unique morphism $h: C^\bullet \rightarrow C'^\bullet$ which gives a morphism of triangles (20).*

2) *For a distinguished triangle*

$$A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \xrightarrow{w} A^\bullet[1]$$

assume that A^\bullet is a complex such that $H^i(A^\bullet)$ are finite-dimensional \mathbb{Q} -vector spaces and C^\bullet is an almost perfect complex. Then the cone of u is unique up to a unique isomorphism.

PROOF. In this situation, by Lemma A.1, the group $\text{Hom}_{\mathbf{D}(\mathbb{Z})}(C^\bullet, C'^\bullet)$ has no non-trivial divisible subgroups, and $\text{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet[1], C'^\bullet)$ is divisible. This means that there are no nontrivial homomorphisms $\text{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet[1], C'^\bullet) \rightarrow \text{Hom}_{\mathbf{D}(\mathbb{Z})}(C^\bullet, C'^\bullet)$, and we can apply Lemma A.2. \square

LEMMA A.4. *Suppose that A^\bullet and B^\bullet are almost of cofinite type in the sense of Definition 1.1. Then a morphism $f: A^\bullet \rightarrow B^\bullet$ is torsion (i.e. a torsion element in the group $\text{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet, B^\bullet)$, i.e. $f \otimes \mathbb{Q} = 0$) if and only if the morphisms $H^i(f): H^i(A^\bullet) \rightarrow H^i(B^\bullet)$ are torsion; that is, they are trivial on the maximal divisible subgroups:*

$$(H^i(f)_{div}: H^i(A^\bullet)_{div} \rightarrow H^i(B^\bullet)_{div}) = 0.$$

PROOF. Under our assumptions, the groups $H^i(A^\bullet)$ and $H^{i-1}(B^\bullet)$ appearing in (18) are of cofinite type. We calculate that in this case, $\text{Ext}(H^i(A^\bullet), H^{i-1}(B^\bullet))$ is finite.

For $i \gg 0$, the groups $H^i(A^\bullet)$ and $H^{i-1}(B^\bullet)$ are finite 2-torsion, and therefore $\text{Ext}(H^i(A^\bullet), H^{i-1}(B^\bullet))$ is finite 2-torsion as well. It follows that the whole product $\prod_{i \in \mathbb{Z}} \text{Ext}(H^i(A^\bullet), H^{i-1}(B^\bullet))$ is of the form $G \oplus T$, where G is finite and T is (possibly infinite) 2-torsion. We have therefore $(G \oplus T) \otimes \mathbb{Q} = 0$.

Similarly, the group $\prod_{i \in \mathbb{Z}} \text{Hom}(H^i(A^\bullet), H^i(B^\bullet))$ consists of some part of the form $\widehat{\mathbb{Z}}^{\oplus r} \oplus G$, where G is finite, and some possibly infinite 2-torsion part, which is killed by tensoring with \mathbb{Q} . It follows from (18) that there is an isomorphism

$$\begin{aligned} \text{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet, B^\bullet) \otimes \mathbb{Q} &\cong \prod_{i \in \mathbb{Z}} \text{Hom}(H^i(A^\bullet), H^i(B^\bullet)) \otimes \mathbb{Q}, \\ f \otimes \mathbb{Q} &\mapsto (H^i(f) \otimes \mathbb{Q})_{i \in \mathbb{Z}}. \end{aligned} \quad \square$$

LEMMA A.5. *If A^\bullet is a complex of \mathbb{Q} -vector spaces and B^\bullet is a complex almost of cofinite type in the sense of Definition 1.1, then there is an isomorphism of abelian groups*

$$\begin{aligned} \text{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet, B^\bullet) &\xrightarrow{\cong} \prod_{i \in \mathbb{Z}} \text{Hom}(H^i(A^\bullet), H^i(B^\bullet)), \\ f &\mapsto (H^i(f))_{i \in \mathbb{Z}}. \end{aligned}$$

PROOF. In the formula (18), if $H^i(A^\bullet)$ are \mathbb{Q} -vector spaces and $H^{i-1}(B^\bullet)$ are groups of cofinite type, then the term $\text{Ext}(H^i(A^\bullet), H^{i-1}(B^\bullet))$ vanishes. \square

B Cohomology with compact support

For any arithmetic scheme $f: X \rightarrow \text{Spec } \mathbb{Z}$ there exists a **Nagata compactification** [6, 7] (see also [1, Exposé XVII])

$$\begin{array}{ccc} X & \xleftarrow{j} & \mathfrak{X} \\ & \searrow f & \swarrow g \\ & & \text{Spec } \mathbb{Z} \end{array}$$

where j is an open immersion and g is a proper morphism.

DEFINITION B.1. Let X be an arithmetic scheme and let \mathcal{F} be an abelian torsion sheaf on $X_{\text{ét}}$. Then one defines the **cohomology with compact support** of \mathcal{F} via the complex

$$R\Gamma_c(X_{\text{ét}}, \mathcal{F}) := R\Gamma(\mathfrak{X}_{\text{ét}}, j_! \mathcal{F}).$$

For torsion sheaves, this does not depend on the choice of $j: X \hookrightarrow \mathfrak{X}$, but here we would like to fix this choice in order to compare cohomology with compact support on $X_{\text{ét}}$ with the singular cohomology with compact support on $X(\mathbb{C})$.

Comparison with the analytic cohomology

DEFINITION B.2. Given a Nagata compactification $j: X \hookrightarrow \mathfrak{X}$, we consider the corresponding open immersion $j(\mathbb{C}): X(\mathbb{C}) \rightarrow \mathfrak{X}(\mathbb{C})$, and for a sheaf \mathcal{F} on $X(\mathbb{C})$ we define

$$\Gamma_c(X(\mathbb{C}), \mathcal{F}) := \Gamma(\mathfrak{X}(\mathbb{C}), j(\mathbb{C})_! \mathcal{F}).$$

Similarly, for a $G_{\mathbb{R}}$ -equivariant sheaf on $X(\mathbb{C})$ we define

$$\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathcal{F}) := \Gamma(G_{\mathbb{R}}, \mathfrak{X}(\mathbb{C}), j(\mathbb{C})_! \mathcal{F}).$$

The canonical reference for the comparison between étale and singular cohomology is [1, Exposé XI, §4], so we borrow some definitions and notations from there. Let X be an arithmetic scheme.

1. The base change from $\text{Spec } \mathbb{Z}$ to $\text{Spec } \mathbb{C}$ gives us a morphism of sites

$$\gamma: X_{\mathbb{C}, \text{ét}} \rightarrow X_{\text{ét}}.$$

2. Let X_{cl} be the site of étale maps $f: U \rightarrow X(\mathbb{C})$. A covering family in X_{cl} is a family of maps $\{U_i \rightarrow U\}$ such that U is the union of images of U_i .

(We recall that in the analytic topology, $f: U \rightarrow X(\mathbb{C})$ is **étale** if it is a *local on the source homeomorphism*: for each $u \in U$ there exists an open neighborhood $u \ni V$ such that $f|_V: V \rightarrow f(V)$ is a homeomorphism.)

Since the inclusion of an open subset $U \subset X(\mathbb{C})$ is an étale map, we have a fully faithful functor $X(\mathbb{C}) \subset X_{cl}$, and the topology on $X(\mathbb{C})$ is induced by the topology on X_{cl} . This gives us a morphism of sites $\delta: X_{cl} \rightarrow X(\mathbb{C})$, which by the comparison lemma [1, Exposé III, Théorème 4.1] induces an equivalence of the corresponding categories of sheaves

$$\delta_*: \mathbf{Sh}(X_{cl}) \rightarrow \mathbf{Sh}(X(\mathbb{C})).$$

3. A morphism of schemes $f: X'_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$ over $\text{Spec } \mathbb{C}$ is étale if and only if the map $f(\mathbb{C}): X'(\mathbb{C}) \rightarrow X(\mathbb{C})$ is étale [20, Exposé XII, Proposition 3.1], and therefore the functor $X'_{\mathbb{C}} \rightsquigarrow X'(\mathbb{C})$ gives us a morphism of sites

$$\epsilon: X_{cl} \rightarrow X_{\mathbb{C}, \text{ét}}.$$

DEFINITION B.3. We define the functor

$$\alpha^*: \mathbf{Sh}(X_{\text{ét}}) \rightarrow \mathbf{Sh}(G_{\mathbb{R}}, X(\mathbb{C}))$$

via the composition

$$\mathbf{Sh}(X_{\text{ét}}) \xrightarrow{\gamma^*} \mathbf{Sh}(X_{\mathbb{C}, \text{ét}}) \xrightarrow{\epsilon^*} \mathbf{Sh}(X_{cl}) \xrightarrow[\simeq]{\delta_*} \mathbf{Sh}(X(\mathbb{C}))$$

As we start from a scheme over $\text{Spec } \mathbb{Z}$ and base change to $\text{Spec } \mathbb{C}$, the resulting sheaf on $X(\mathbb{C})$ is equivariant with respect to the complex conjugation, hence an object in $\mathbf{Sh}(G_{\mathbb{R}}, X(\mathbb{C}))$. For the definition of equivariant sheaves, we refer to the introduction.

LEMMA B.4. α^* preserves colimits.

PROOF. α^* is the composition of the inverse image functors γ^* and ϵ^* (which are left adjoint) and an equivalence δ_* . \square

PROPOSITION B.5. Given a sheaf \mathcal{F} on $X_{\acute{e}t}$, there exists a natural morphism

$$\Gamma(X_{\acute{e}t}, \mathcal{F}) \rightarrow \Gamma(G_{\mathbb{R}}, X(\mathbb{C}), \alpha^* \mathcal{F}),$$

and similarly, for cohomology with compact support,

$$\Gamma_c(X_{\acute{e}t}, \mathcal{F}) \rightarrow \Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \alpha^* \mathcal{F}).$$

PROOF. If $j: X \hookrightarrow \mathfrak{X}$ is a Nagata compactification, we have the corresponding compactification $j(\mathbb{C}): X(\mathbb{C}) \hookrightarrow \mathfrak{X}(\mathbb{C})$. The extension by zero morphism $j(\mathbb{C})_!: \mathbf{Sh}(X(\mathbb{C})) \rightarrow \mathbf{Sh}(\mathfrak{X}(\mathbb{C}))$ restricts to the subcategory of $G_{\mathbb{R}}$ -equivariant sheaves: if \mathcal{F} is a $G_{\mathbb{R}}$ -equivariant sheaf on $X(\mathbb{C})$, then $j(\mathbb{C})_! \mathcal{F}$ is a $G_{\mathbb{R}}$ -equivariant sheaf on $\mathfrak{X}(\mathbb{C})$. From the definition of α^* , we see that there is a commutative diagram

$$\begin{array}{ccc} \mathbf{Sh}(X_{\acute{e}t}) & \xrightarrow{\alpha^*} & \mathbf{Sh}(G_{\mathbb{R}}, X(\mathbb{C})) \\ j_! \downarrow & & \downarrow j(\mathbb{C})_! \\ \mathbf{Sh}(\mathfrak{X}_{\acute{e}t}) & \xrightarrow{\alpha_{\mathfrak{X}}^*} & \mathbf{Sh}(G_{\mathbb{R}}, \mathfrak{X}(\mathbb{C})) \end{array}$$

—this diagram commutes for representable étale sheaves, and then every étale sheaf is a colimit of representable sheaves, and α^* , $j_!$, $\alpha_{\mathfrak{X}}^*$, $j(\mathbb{C})_!$ preserve colimits, as left adjoints.

The morphism in question is given by

$$\begin{aligned} \Gamma_c(X_{\acute{e}t}, \mathcal{F}) &:= \Gamma(\mathfrak{X}_{\acute{e}t}, j_! \mathcal{F}) \rightarrow \Gamma(G_{\mathbb{R}}, \mathfrak{X}(\mathbb{C}), \alpha_{\mathfrak{X}}^* j_! \mathcal{F}) \\ &= \Gamma(G_{\mathbb{R}}, \mathfrak{X}(\mathbb{C}), j(\mathbb{C})_! \alpha^* \mathcal{F}) =: \Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \alpha^* \mathcal{F}). \quad \square \end{aligned}$$

The morphism α is also discussed in [9, Appendix A], but Flach and Morin work with proper schemes; the above remarks are to make sure that everything works fine for compactifications.

Modified étale cohomology

Here we briefly review the **modified étale cohomology with compact support** $R\widehat{\Gamma}_c(X_{\acute{e}t}, -)$. It was introduced by Th. Zink in [21, Appendix 2] for the case of number rings $X = \text{Spec } \mathcal{O}_{K,S}$, and it is also discussed in [32, §II.2]. The general definition for $X \rightarrow \text{Spec } \mathbb{Z}$ is treated in [9, §6.7] and [17, §2].

Thanks to the Leray spectral sequence $R\Gamma(\mathfrak{X}_{\acute{e}t}, -) \cong R\Gamma(\text{Spec } \mathbb{Z}_{\acute{e}t}, -) \circ Rg_*$, we have

$$R\Gamma_c(X_{\acute{e}t}, \mathcal{F}) := R\Gamma(\mathfrak{X}_{\acute{e}t}, j_! \mathcal{F}) \cong R\Gamma((\text{Spec } \mathbb{Z})_{\acute{e}t}, Rf_! \mathcal{F}), \quad \text{where } Rf_! \mathcal{F} := Rg_* j_! \mathcal{F}.$$

First we recall that for a finite group G and a G -module A the corresponding group cohomology $H^i(G, A)$ (resp. Tate cohomology $\widehat{H}^i(G, A)$) can be defined in terms of resolutions P_\bullet (resp. complete resolutions \widehat{P}_\bullet) of \mathbb{Z} by free $\mathbb{Z}G$ -modules (see e.g. [5, Chapter VI]). More generally, if A^\bullet is a bounded (cohomological) complex of G -modules, we obtain a *double complex* of abelian groups $\text{Hom}^{\bullet\bullet}(P_\bullet, A^\bullet)$ (resp. $\text{Hom}^{\bullet\bullet}(\widehat{P}_\bullet, A^\bullet)$), and it makes sense to define the corresponding **group hypercohomology** (resp. **Tate hypercohomology**) via the complexes

$$R\Gamma(G, A^\bullet) := \text{Tot}^\oplus(\text{Hom}^{\bullet\bullet}(P_\bullet, A^\bullet)), \quad R\widehat{\Gamma}(G, A^\bullet) := \text{Tot}^\oplus(\text{Hom}^{\bullet\bullet}(\widehat{P}_\bullet, A^\bullet)).$$

Now if \mathcal{F} is an abelian sheaf on $(\text{Spec } \mathbb{Z})_{\acute{e}t}$, then the corresponding **modified cohomology with compact support** is characterized by the distinguished triangle

$$R\widehat{\Gamma}_c((\text{Spec } \mathbb{Z})_{\acute{e}t}, \mathcal{F}) \rightarrow R\Gamma((\text{Spec } \mathbb{Z})_{\acute{e}t}, \mathcal{F}) \rightarrow R\widehat{\Gamma}(G_{\mathbb{R}}, v^* \mathcal{F}) \rightarrow R\widehat{\Gamma}_c((\text{Spec } \mathbb{Z})_{\acute{e}t}, \mathcal{F})[1]$$

Here $v: \text{Spec } \mathbb{R} \rightarrow \text{Spec } \mathbb{Z}$ is the canonical morphism, and $v^* \mathcal{F}$ is the corresponding sheaf on $(\text{Spec } \mathbb{R})_{\acute{e}t}$, which can be viewed as a $G_{\mathbb{R}}$ -module by [1, Exposé VII, 2.3], and $R\widehat{\Gamma}(G_{\mathbb{R}}, v^* \mathcal{F})$ denotes the corresponding Tate cohomology.

In general, given an arithmetic scheme $X \rightarrow \text{Spec } \mathbb{Z}$ and a torsion abelian sheaf \mathcal{F} on $X_{\acute{e}t}$, we choose a Nagata compactification as above and set

$$R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathcal{F}) := R\widehat{\Gamma}_c((\text{Spec } \mathbb{Z})_{\acute{e}t}, Rf_! \mathcal{F}).$$

We have a natural morphism

$$R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathcal{F}) \rightarrow R\Gamma_c(X_{\acute{e}t}, \mathcal{F}),$$

which is an isomorphism if $X(\mathbb{R}) = \emptyset$. In general, Tate cohomology $\widehat{H}^i(G_{\mathbb{R}}, -)$ is annihilated by multiplication by $2 = \#G_{\mathbb{R}}$, and therefore $\widehat{H}^i_c(X_{\acute{e}t}, \mathcal{F}) \rightarrow H^i_c(X_{\acute{e}t}, \mathcal{F})$ has 2-torsion kernel and cokernel.

For canonicity and functoriality, I refer to [17, §2].

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